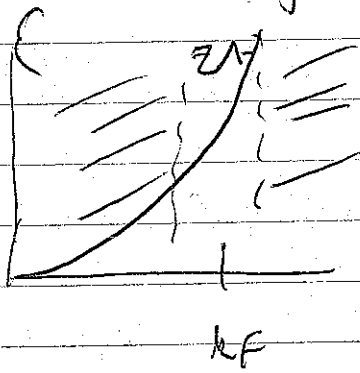
[J in tight binding model is normalized differently - has dimension of energy]

7/12 - PS 3 posted

- in typical experiments $J\nu \ll 1$ so we can use perturbation theory, improved by the RG

- we ~~eliminate~~ integrate out momentum components $\psi, \psi(k, \omega, \sigma)$ with large $|\epsilon_{\vec{k}}|$ & calculate effective Hamilton & action

- for small $J\nu$, changes are initially small, but as width of remaining energy band $\to 0$ they start to blow up



- ~~at small Λ we~~

- at small Λ we can linearize dispersion relation: $\mathcal{E}(k) \approx v_F(k - k_F)$

and model becomes approximately equivalent to a relativistic model, as we will discuss

N.B - we do not integrate out $\vec{r}$ in weak coupling RG, only some momentum components $\Lambda < \omega$, ~~or weak coupling RG~~

- in a sense, we do integrate out $\vec{r}$ in strong coupling RG, to be discussed later

- it is possible to write a path integral representation of spin using spin coherent states, but this isn't really necessary or useful, so I will use a hybrid approach: path integral for fermions

but quantum trace for $\vec{S}$

ie resolution of identity used is

$$\mathbb{I} = \int \prod_{\alpha,\vec{k}} d\bar{\psi}_\alpha(\vec{k})\, d\psi_\alpha(\vec{k})\, \exp\Big[- \sum_{\alpha,\vec{k}} \bar{\psi}_\alpha(\vec{k})\psi_\alpha(\vec{k}) \Big] \prod_{\alpha,\vec{k}} |\psi_\alpha(\vec{k})\rangle$$

$$\mathbb{I} = \prod_{\alpha,\vec{k}} d\bar{\psi}_\alpha(\vec{k})\, d\psi_\alpha(\vec{k})\, e^{-\bar{\psi}_\alpha(\vec{k})\psi_\alpha(\vec{k})} |\psi_\alpha(\vec{k})\rangle\langle\bar{\psi}_\alpha(\vec{k})|$$

$$\otimes \sum_\sigma |\sigma\rangle\langle\sigma|$$

where $\sigma = \uparrow, \downarrow$ labels impurity spin states

- we insert $\mathbb{I}$ N-times to get rid of fermion operators - spin operators remain

$$Z \to \sum_\sigma \langle\sigma| \prod_{i=1}^{N} [d\bar{\psi}(\tau_n)\, d\psi(\tau_n)]$$

$$\exp\Big[\sum_{\vec{k},\alpha} \bar{\psi}_{\vec{k}}(\tau_i)\big(\psi_{\vec{k}}(\tau_i) - \psi_{\vec{k}}(\tau_{i-1})\big) - \varepsilon H(\bar{\psi}(\tau_{i+1}), \psi(\tau_i), \vec{S}) \Big] |\sigma\rangle$$

N.B. - ℓ re-exponentiates $(-\varepsilon H)$ factors but ℓ didn't replace product

of exponentials by exponential of sum

- this wouldn't be legitimate because because spin operators don't commute with each other

- we can take care of this by giving spin operators an imaginary time label $\vec{S}(\tau_i)$ and writing

$$Z = \mathcal{C}, \mathcal{T} \int [d\bar{\psi} d\psi] \exp\left[- \int_0^\beta d\tau \, \bar{\psi} \frac{d}{d\tau} \psi + H(\bar{\psi}(\tau), \psi(\tau), \tau) \right]$$

(where I suppressed $\vec{k}$, α labels on $\psi_\alpha(\vec{k}, \tau)$)

$\mathcal{T}$ is time ordering symbol (for spin operators only) - latest on left

- only with time-ordering can we replace product of exponentials by exponential of sum

- W/ D- spin operators don't actually depend on time but we must order them by time

$$T\left[S^x(\tau_1)\, S^x(\tau_2)\right] = \left(S^x\right)^2 = \frac{1}{4}\, I$$

$$T\left[S^x(\tau_1)\, S^y(\tau_1)\right] = S^x S^y = \frac{i}{2} S^z, \quad \tau_1 > \tau_2$$

$$= S^y S^x = -\frac{i}{2} S^z, \quad \tau_1 < \tau_2.$$

- we use bosonic sign convention for time-ordering for spin operators

$$T\left[S^a(\tau_1)\, S^b(\tau_2)\right] = \frac{1}{4}\delta^{ab}\, I$$

$$+ \mathrm{sgn}(\tau_1 - \tau_2)\ \frac{i}{2}\epsilon^{abc} S^c$$

- we will Taylor expand H_{INT} term in exponential to do perturbation theory in J
- we must time-order S^a's in each term
- note that, in interaction picture,

$$\vec{S}(\tau) = \vec{S}$$ since H_0 is independent of $\vec{S}$

- we now calculate renormalization of S_{eff} by integrating out modes far from fermi surface

- following Shankar, let $\psi_>, \bar{\psi}_>$ represent modes with large $|\epsilon_{\vec{k}}|$ and $\psi_<, \bar{\psi}_<$ modes with small $|\epsilon_{\vec{k}}|$

- in general we will have an initial cut-off Λ and a reduced cut-off Λ' so $\Lambda' < |\epsilon_{\vec{k}}| < D$ for $\psi_>$ and $|\epsilon_{\vec{k}}| < D'$ for $\psi_<$

$$\int [d\bar{\psi} d\psi] = \int [d\bar{\psi}_< d\psi_<] [d\bar{\psi}_> d\psi_>]$$

- do integral over $\psi_>, \bar{\psi}_>$ and interpret result as a renormalized $S_{eff}(\bar{\psi}_<, \psi_<)$

$$S_0 = -\int \frac{d\bar{\psi} d\psi}{} \sum_{\vec{k},\alpha} \bar{\psi}_\alpha(\vec{k}) \left[\frac{\partial}{\partial \tau} + \epsilon_{\vec{k}} \right] \psi_\alpha(\vec{k})$$

$$= S_{0>} + S_{0<}$$

with $S_{int} = -\frac{J}{V} \int d\tau \sum_{\vec{k}_1, \vec{k}_2} \bar{\psi}(\vec{k}_2, \tau) \frac{\vec{\sigma}}{2} \psi(\vec{k}_1, \tau) \cdot \vec{S}(\tau)$

also has cross-terms

[sum over spin indices implied]

It can be seen that expanding to 1st order in S_{INT} gives to renormalization of S_{eff}, using $\mathrm{tr}\,\vec{\sigma} = 0$

- we will work at $T=0$, $\beta \to \infty$
- it is convenient to let T integral run from $-\frac{\beta}{2}$ to $\frac{\beta}{2}$, using periodicity, so it runs from $-\infty$ to ∞ at $\beta \to \infty$
- for Kondo model it is convenient to work in time-domain, not frequency
- 2nd order expansion of $e^{+S_{INT}}$ is

$$\frac{1}{2}\left(\frac{J}{V}\right)^2 \int_{-\infty}^{\infty} d\tau_1\, d\tau_2 \sum_{\substack{\vec{k}_1,\vec{k}_1',\\ \vec{k}_2,\vec{k}_2'}} T\Big[\psi^\dagger(\vec{k}_1,\tau_1)\,\frac{\vec{\sigma}}{2}\,\psi(\vec{k}_1',\tau_1)\cdot\vec{S}(\tau_1)$$

$$\psi^\dagger(\vec{k}_2,\tau_2)\,\frac{\vec{\sigma}}{2}\,\psi(\vec{k}_2',\tau_2)\cdot\vec{S}(\tau_2)\Big]$$

[sum over spin indices suppressed]

- if all 4 wave-vectors are in $<$ range we don't need to bother expanding

- if all k are in $\tilde{k}$ range, integral just gives a constant, independent of $\psi_<, \bar{\psi}_<$ - shifts ground state energy

- interesting case has either $\vec{k}_1', \vec{k}_2 \in >$

or $\vec{k}_1, \vec{k}_2' \in >$ (only)

- then we do Grassmann integrals

$$\langle \psi_\beta(\vec{k}_1', \tau_1)\, \bar{\psi}_\delta(\vec{k}_2, \tau_2)\rangle \text{ or}$$

$$\langle \bar{\psi}_\alpha(\vec{k}_1, \tau_1)\, \psi_\delta(\vec{k}_2', \tau_2)\rangle$$

- these are simply ~~massless~~ Matsubara GF's at $\beta \to \infty$ which give

$$\langle \psi(\vec{k}_2, \tau_1)\, \bar{\psi}(\vec{k}_1, \tau_2)\rangle$$

$$= \left[\theta(\tau_1-\tau_2)\langle c_{\vec{k}}(\vec{k}^+\rangle - \theta(\tau_2-\tau_1)\langle c_{\vec{k}}^+(\vec{k}\rangle\right] e^{(\tau_2-\tau_1)\epsilon_{\vec{k}}}$$

$$= \left[\theta(\tau_1-\tau_2)\,\theta(+\epsilon_{\vec{k}}) - \theta(\tau_2-\tau_1)\,\theta(-\epsilon_{\vec{k}})\right] e^{(\tau_2-\tau_1)\epsilon_{\vec{k}}}$$

NB - exponent always <0 - can replace by

$$e^{-|\tau_2-\tau_1|\,|\epsilon_{\vec{k}}|}$$

Term with $\vec{k}_1, \vec{k}_2 \in <$ gives

$$\frac{1}{2}\left(\frac{J}{V}\right)^2 \sum_{\substack{\vec{k}_1,\vec{k}_2 \in < \\ ab}} \int d\tau_1 d\tau_2 \; \bar{\psi}(\vec{k}_1,\tau_1)\frac{\sigma^a}{2}\frac{\sigma^b}{2}\psi(\vec{k}_2,\tau_2)$$

$$T\left[\xi^a(\tau_1)\xi^b(\tau_2)\right]$$

$$\cdot\left[\theta(\tau_1-\tau_2)\theta(\epsilon_{k_2}) - \theta(\tau_2-\tau_1)\theta(-\epsilon_{k_2})\right]\exp\left[(\tau_2-\tau_1)\epsilon_{k_2}\right]$$

- here I used fact that $\langle \psi_\beta(\vec{k}_1,\tau_1)\psi_\delta(\vec{k}_2,\tau_2)\rangle \propto \delta_{\beta\delta}$ gives

$$\sum \bar{\psi}_\alpha \sigma^a_{\alpha\beta}\delta_{\beta\delta}\sigma^b_{\delta\delta}\psi_\delta = \bar{\psi}\sigma^a\sigma^b\psi$$

- now consider 2^{nd} term with $\vec{k}_1, \vec{k}_2 \in >$

- I can anti-commute Remaining $\psi(\vec{k}_1',\tau_1^b)\;\bar{\psi}(\vec{k}_2,\tau_2)$ into order $\bar{\psi}\psi$ at cost of a minus sign

~~if~~ I relabel integration and summation variables $(\tau_1 \leftrightarrow \tau_2)$, $(\vec{k}_2 \leftrightarrow \vec{k}_1)$, $(\vec{k}_2' \leftrightarrow \vec{k}_1')$, $(a \leftrightarrow b)$

- I get identical expression except order of $\xi^a(\tau_1)$, $\xi^b(\tau_2)$ switched

[i.e. $\sum \int \bar{\psi}(\tau_2)\sigma^b\sigma^a\psi(\tau_1)\,T\,\xi^a(\tau_1)\xi^b(\tau_2) \to \sum \int \bar{\psi}(\tau_1)\sigma^a\sigma^b\psi(\tau_2)\,T\,\xi^b(\tau_2)\xi^a(\tau_1)$]

adding 2 terms together gives a factor 1

$T[S^a(\tau_1), S^b(\tau_2)] = \mathrm{sgn}(\tau_1-\tau_2)\, i\epsilon^{abc} S^c$

- NB only non-zero because S^a, S^b don't commute

- now simplify trace over spin indices in

fermion using

$$\sum_{a,b} \epsilon^{abc}\sigma^a\sigma^b = 2i\sigma^c$$

- 2^{nd} order term becomes

$$-\left(\frac{J}{\bar{V}}\right)^2 \int_{-\infty}^{\infty} d\tau_1 d\tau_2 \sum_{\vec{k}_1,\vec{k}_1' \in <} \bar{\psi}(\vec{k}_1,\tau_1)\frac{\vec{\sigma}}{2}\psi(\vec{k}_1',\tau_2)\cdot\vec{S}$$

$$\frac{1}{2}\sum_{k_2\in >}\left[\theta(\tau_1-\tau_2)\theta(\epsilon_{k_2})-\theta(\tau_2-\tau_1)\theta(-\epsilon_{k_2})\right] e^{(\tau_2-\tau_1)\epsilon_{k_2}}$$

$$\mathrm{sgn}(\tau_1-\tau_2)$$

[minus sign from i^2]

- this looks like an $O(J^2)$ correction to J

- however it is a retarded interaction

involving $\bar{\psi}(\tau_1)\ \psi(\tau_2)$ at different times

- $\int d\tau_1 d\tau_2\, \bar{\psi}(\tau_1)\psi(\tau_2) f(\tau_1-\tau_2)$

- we will Taylor expand $\psi(\tau_2) = \psi(\tau_1) + (\tau_2-\tau_1)\frac{d\psi}{d\tau} + \cdots$

- can integrate over τ_2 term by term
- in this way we generate a series of terms in left which are instantaneous but involve time derivatives of Ψ
- keep only non-derivative term for now
- other ones, as we will see, correspond to irrelevant operators

let $\tau_2 - \tau_1 = \tau'$

- 2nd order term becomes

$$-\left(\frac{J}{V}\right)^2 \int d\tau \sum_{k,k' \in <} \Psi(\vec{k},\tau) \frac{\vec{\sigma}}{2} \Psi(\vec{k}',\tau) \cdot \vec{S}$$

$$\frac{1}{2}\int d\tau' \sum_{\vec{k}_2 \in >} \Big[\theta(-\tau')\theta(\epsilon_{k_2}) - \theta(+\tau')\theta(-\epsilon_{k_2}) \Big] \mathrm{sgn}(\omega - \tau') \exp\left[\tau' \epsilon_{k_2}\right]$$

- integral over $d\tau'$ gives

$$\alpha \int_{-\infty}^{0} d\tau' e^{\tau'\epsilon} + \theta(-\epsilon) \int_{0}^{\infty} d\tau' e^{\tau'\epsilon}$$

$$= \frac{\theta(\epsilon)}{\epsilon} + \frac{\theta(-\epsilon)}{-\epsilon} = \frac{1}{|\epsilon|}$$

learn

$$-\frac{J^2}{V}\sum_{\vec{k},\vec{k}'\in<}\int d\tau\ \bar{\psi}(\vec{k},\tau)\,\vec{\sigma}\,\psi(\vec{k}',\tau)\cdot\vec{S}$$

$$\frac{1}{2}\frac{1}{V}\sum_{\vec{k}_2\in>}\frac{1}{|\epsilon_{\vec{k}_2}|}$$

- we ~~interpret~~ Re-exponentiate

$$e^{\delta S_{INT}} = 1 + \delta S_{INT}$$

$$\delta S_{INT} = -\int d\tau\ \delta H_{INT}$$

- correction to S_{INT} has form of a correction to Ising coupling

$$\delta J = J^2\cdot\frac{1}{V}\frac{1}{2}\sum_{\vec{k}_2\in>}\frac{1}{|\epsilon_{\vec{k}_2}|}$$

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take $V\to\infty$

$$\delta J = J^2\,\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{|\epsilon_{\vec{k}}|}$$

$\vec{k}\in>$

$$= J^2\times\frac{1}{2}\int d\epsilon\,\frac{\nu(\epsilon)}{|\epsilon|}$$

where $\nu(\epsilon)$ = density of states

- integral is over $D' < \epsilon < D$ and $-D < \epsilon < -D'$

corresponding to region of $>$ modes

as D' is reduced towards 0 [$\ll \epsilon_F$]

$\int T$ diverges since $V(\epsilon)$ can be approximated

by $V(\epsilon_F) = V$

$\int T \simeq J^2 V \ln \frac{D}{D'}, > 0$ and

diverges as $D' \to 0$

[$\ln D$ term is just an estimate

becomes correct if D' is also $\ll \epsilon_F$ - otherwise

there is a finite correction term]

- the RG is usually performed in infinitesimal steps [more powerful that way]

- we consider an infinitesimal reduction

$D' = D^* - \delta D$, $\delta D \ll D$

and we consider infinitesimal change

in Jetf

- when the running cut-off D is near D, the change in coupling constants depends on their values at D

- in previous calculation, we interpret J as $J(D)$ - running coupling constant when cut-off is D

$$\ln \frac{D}{D'} = \ln \frac{D}{D-\delta D} \approx \frac{\delta D}{D}$$

$$\delta J = J^2(D)\, V \frac{\delta D}{D}$$

$$D \frac{dJ}{dD} = -V J^2(D) \quad [\text{change on } D \text{ is } -\delta D]$$

$$\frac{dJ}{d\ln D} = -V J^2$$

if we define dimensionless Kondo coupling

$$\lambda \equiv JV$$

$$\frac{d\lambda}{d\ln D} = -\lambda^2$$

- this verifies that, as expected, $J\nu$ is appropriate def'n of dimensionless coupling constant

more generally if $\frac{d\lambda}{d\ln D} = \beta(\lambda)$ (*)

$\beta(\lambda)$ is called the β-function

$\beta = -\lambda^2$ in this lowest order perturbative calculation

- I will refer to (*) as the RG equation

- a fundamental part of the RG

- equation is easily solved

let λ_0 be $\lambda(D_0)$ for some relatively high energy cut-off D_0 and express $\lambda(D)$ at lower cut-off D in terms of λ_0 and D_0

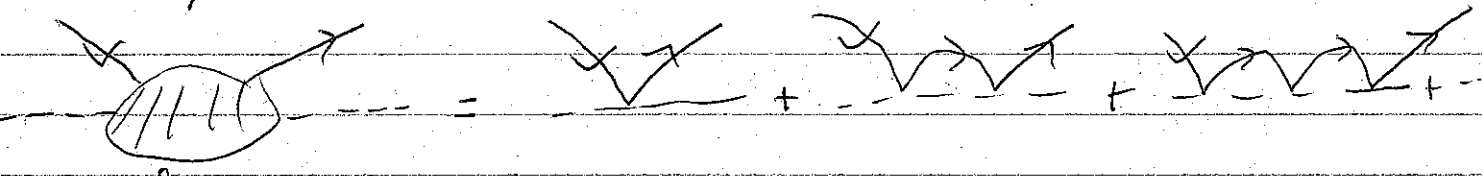
$$\int_{\lambda_0}^{\lambda} \frac{d\lambda}{\lambda^2} = \int_{D_0}^{D} d(\ln D)$$

$$\frac{1}{\lambda_0} - \frac{1}{\lambda} = \ln \frac{D_0}{D}$$

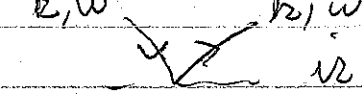
$$\lambda(D) = \frac{\lambda_0}{1 - \lambda_0 \ln \frac{D_0}{D}} \doteq$$

$$\doteq \lambda_0 + \lambda_0^2 \ln \frac{D_0}{D} + \lambda_0^3 \ln^2 \frac{D_0}{D} + \cdots$$

- note that our strictly perturbative calculation gave only first 2 terms, by using RG equation we get complete series
- this corresponds to summing a series of diagrams



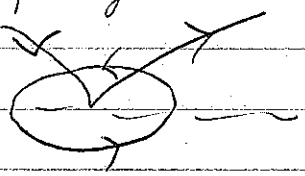
where ---- represents impurity spin,



is Kondo interaction

- but there are other diagrams besides these ones which give higher order terms in

β - function



$$\lambda = \lambda_0 + \lambda_0^2 \ln \frac{D}{D'} + \lambda_0^3 \ln^2 \frac{D}{D'} - \frac{1}{2} \lambda_0^3 \ln \frac{D}{D'}$$

- lowest order β-function corresponds to "leading log" approximation $\lambda_0^n \left(\ln \frac{D}{D'}\right)^{n-1}$

$\lambda_0^3 \ln \frac{D}{D'}$ gives $O(\lambda^3)$ term in β function

$$\frac{d\lambda}{d \ln D} = -\lambda_0^2 - 2\lambda_0^3 \ln \frac{D}{D'} + \frac{1}{2}\lambda_0^3$$

$$\approx -\left(\lambda_0 + \lambda_0^2 \ln \frac{D}{D'}\right)^2 + \frac{1}{2}\lambda_0^3$$

$$\approx -\lambda^2 + \frac{1}{2}\lambda^3$$

$$\beta(\lambda) = -\lambda^2 + \frac{1}{2}\lambda^3 + O(\lambda^4)$$

- if we knew $\beta(\lambda)$ exactly, we could determine $\lambda(D)$ at any scale D

[ignoring "irrelevant operators"]

- but we don't

- our lowest order result, *, is only valid over range of D where $|\lambda(D)| \ll 1$ so we can ignore higher order corrections in β-function

* $\lambda > 0$ (AF), $\lambda < 0$ (F) are very different

if $\lambda_0 < 0$, $\lambda(D) = \frac{\lambda_0}{1 - \lambda_0 \ln \frac{D_0}{D}} = \frac{\lambda_0}{1 + |\lambda_0| \ln \frac{D_0}{D}}$

keeps getting smaller as D is reduced

- so if bare coupling λ_0 is small and negative higher order terms in β-function never become important, and we see that $\lambda(D) \to 0$ as $D \to 0$, $\lambda(D) \sim \frac{-1}{\ln D_0/D}$

- on the other hand, if $\lambda_0 > 0$, as appears to be the case experimentally for most

types of impurities and metals, then $\lambda(D)$ grows as D is reduced and we always eventually reach a cut-off scale D where higher-order terms become important

- NB- knowing cubic term doesn't help very much usually, because at low enough D that it is important, $\lambda(D) \sim 1$, quartic & higher terms are also important

- basic idea of RG-improved perturbation theory is to calculate some physical quantity, eg. resistivity, to lowest order in λ, then replace λ by its renormalized value at the energy scale E that characterizes the quantity we are trying to calculate

- this corresponds to summing up an

∞ series of corrections to lowest order

Calculation in leading log approx.

[will return to some fine points about this later]

- for F case this is a very powerful technique - even though perturbation theory is apparently breaken down:

$\lambda_0 + \lambda_0^2 \ln \frac{D_0}{D} + \lambda_0^3 \ln^2 \frac{D_0}{D} + \cdots$ we sum series to get a small quantity

$$\frac{\lambda_0}{1 + |\lambda_0| \ln \frac{D_0}{D}} \to 0$$

- also useful for $\lambda > 0$ but less so

- how we can rely use RG improved perturbation theory above energy scale ~~were~~ where $\lambda(\epsilon) \sim 1$