The Spherical Harmonics and angular momentum operators

$$\frac{-\hbar\nabla^2}{2m}\Psi + V(r)\Psi = E\Psi$$

Schrodinger Eqn

↑	↑	↑
Kinetic energy	Potential energy	Total energy

$$\frac{-\hbar\nabla^2}{2m}\Psi + V(r)\Psi = E\Psi$$

Schrodinger Eqn

$$\Psi = R(r)Y(\theta, \phi)$$

separation ansatz

$$\frac{-\hbar\nabla^2}{2m}\Psi + V(r)\Psi = E\Psi$$

Schrodinger Eqn

$$\Psi = R(r)Y(\theta,\phi)$$

separation ansatz

$$\nabla^2$$

$$\left\{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}[V(r)-E]\right\} = -\frac{1}{Y}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right\}$$

$$= \qquad\qquad =$$

$$l(l+1) \qquad\qquad l(l+1)$$

both sides constant

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)Y$$

$$\frac{-\hbar\nabla^2}{2m}\Psi + V(r)\Psi = E\Psi$$ Schrodinger Eqn

$$\Psi = R(r)Y(\theta, \phi)$$ separation ansatz

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)Y$$ angular equation

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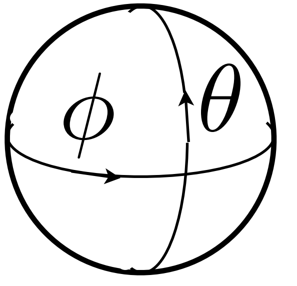

spherical harmonics

$$Y_l^m = \epsilon\sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}}\,e^{im\phi}\,P_l^m(\cos\theta)$$

Condon-Shortley phase

$$\epsilon = (-1)^m \quad m \geq 0$$
$$\epsilon = 1 \quad m < 0$$

orthonormal

$$\left\langle Y_l^m | Y_{l'}^{m'} \right\rangle = \delta_{ll'}\delta_{mm'}$$

$$Y_l^m = \epsilon \sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}} \; e^{im\phi} \; P_l^m(\cos\theta)$$

spherical harmonics

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spherical harmonics

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angular equation

$$-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] = \hbar^2\ l(l+1)Y$$

$$Y_l^m = \epsilon \sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}}\; e^{im\phi}\; P_l^m(\cos\theta)$$

spherical harmonics

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)Y$$

angular equation

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$$\hat{L}^2\, Y = \hbar^2\, l(l+1)Y$$

eigenvalue equation

angular momentum operator written in spherical coordinates (section 4.3.2)

$$Y_l^m = \epsilon\sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}}\; e^{im\phi}\; P_l^m(\cos\theta)$$

spherical harmonics

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)Y$$

angular equation

$$-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] = \hbar^2\, l(l+1)Y$$

angular momentum operator written in spherical coordinates

$$\hat{L}^2\, Y = \hbar^2\, l(l+1)Y$$

eigenvalue equation

also...

$$\frac{\hbar}{i}\frac{\partial}{\partial\phi}Y = \hbar m Y$$

$$Y_l^m = \epsilon \sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}}\; e^{im\phi}\; P_l^m(\cos\theta)$$

spherical harmonics

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)Y$$

angular equation

$$-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] = \hbar^2\, l(l+1)Y$$

angular momentum operator
written in spherical coordinates

$$\hat{L}^2\, Y = \hbar^2\, l(l+1)Y$$

eigenvalue equation

also...

$$\frac{\hbar}{i}\frac{\partial}{\partial\phi}Y = \hbar m Y$$

Lz operator written in spherical coordinates

$$\hat{L}_z\, Y = \hbar m Y$$

eigenvalue equation

$$Y_l^m = \epsilon \sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}}\; e^{im\phi}\; P_l^m(\cos\theta)$$

spherical harmonics

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)Y$$

angular equation

$$-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] = \hbar^2\, l(l+1)Y$$

angular momentum operator written in spherical coordinates

$$\hat{L}^2\, Y = \hbar^2\, l(l+1)Y$$

eigenvalue equation

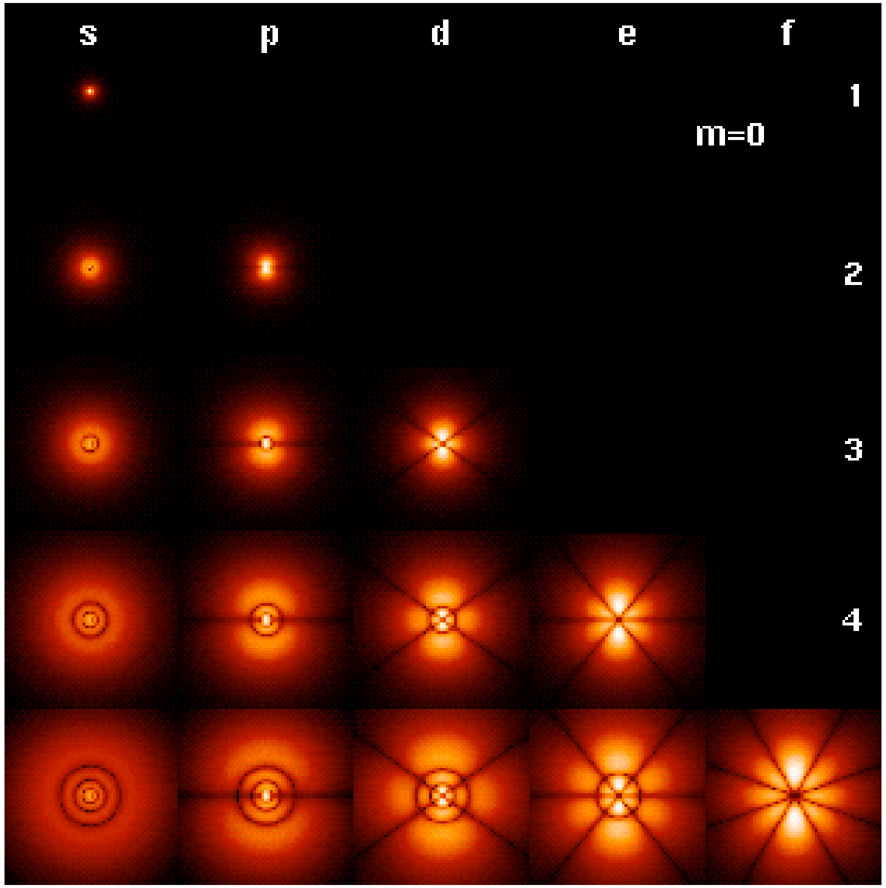
also...

$$\frac{\hbar}{i}\frac{\partial}{\partial\phi}Y = \hbar m Y$$

$$\hat{L}_z\, Y = \hbar m Y$$

Amazing! The beautiful spherical harmonics (Y) are eigenfunctions of L and Lz !

$$\psi_{nlm} = R(r)\, Y(\theta, \phi)$$ hydrogen wave functions



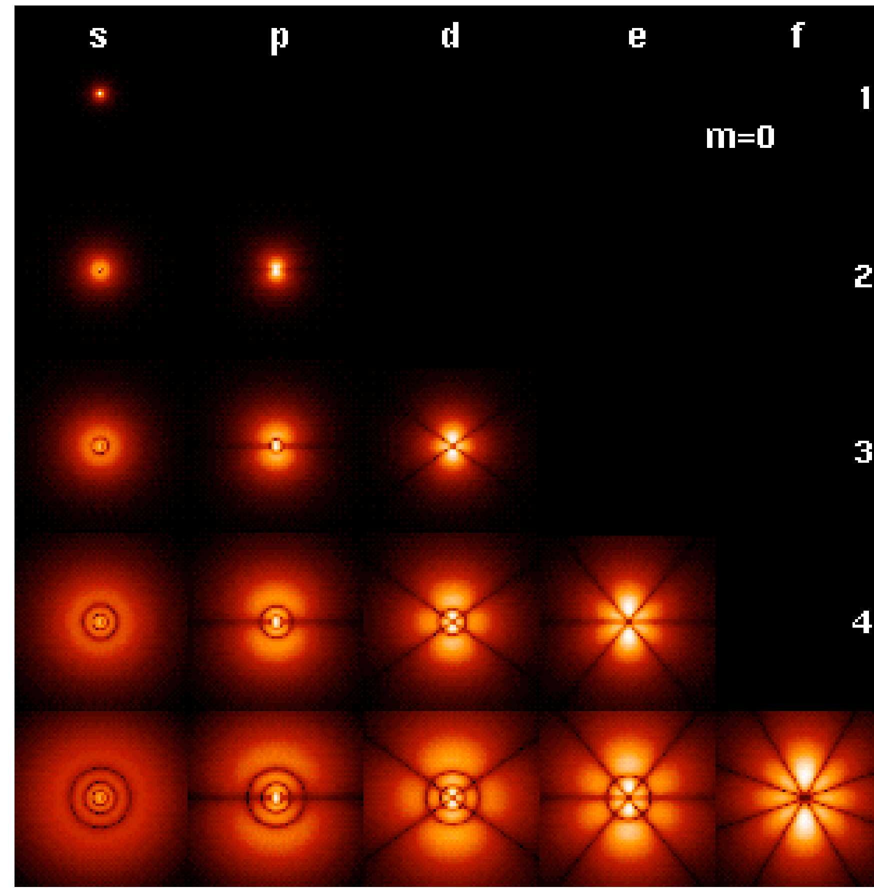
$$\psi_{nlm} = R(r)\, Y(\theta, \phi)$$ hydrogen wave functions

$$\hat{L}^2\, \psi_{nlm} = \hbar^2 l(l+1)\, \psi_{nlm}$$

$$\hat{L}_z\, \psi_{nlm} = \hbar m\, \psi_{nlm}$$

$$\hat{H}\, \psi_{nlm} = E_n\, \psi_{nlm}$$

$$E_n = \frac{E_1}{n^2}$$



simultaneous eigenfunctions of L, Lz, and H.