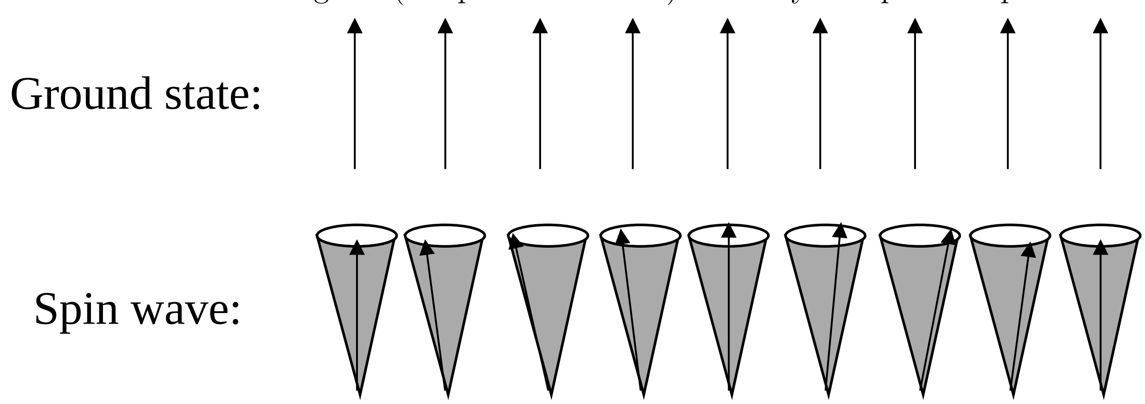
PHYSICS 455
Assignment 5
Due March 25, 2001

1. (Schroeder 7.64) A ferromagnet is a material (like iron) that magnetizes spontaneously, even in the absence of an externally applied magnetic field. This happens because each elementary dipole has a strong tendency to align parallel to its neighbors. At $T = 0$ the magnetization of a ferromagnet has the maximum possible value, with all dipoles perfectly lined up; if there are N atoms, the total magnetization is typically $\approx 2\mu_B N$, where μ_B is the Bohr magneton. At somewhat higher temperatures, the excitations take the form of spin waves, which can be visualized classically as shown in the figure. Like sounds waves, spin waves are quantized: Each wave mode can have only integer multiples of a basic energy unit. In analogy with phonons, we think of the energy units as particles, called magnons. Each magnon reduces the total spin of the system by one unit of $h/2\pi$, and therefore reduces the magnetization by $\approx 2\mu_B$. however, whereas the frequency of a sound wave is inversely proportional to its wavelength, the frequency of a spin wave is proportional to the square of $1/\lambda$ (in the limit of long wavelengths). Therefore, since $\mathcal{E} = hf$ and $p = h/\lambda$ for any "particle," the energy of a magnon is proportional to the square of its momentum. In analogy with the energy-momentum relation for an ordinary nonrelativistic particle, we can write $\mathcal{E} = p^2/2m^*$, where m^* is a constant related to the spin-spin interaction energy and the atomic spacing. For iron, m^* turns out to equal 1.24×10^{-29} kg, about 14 times the mass of an electron. Another difference between magnons and phonons is that each magnon (or spin wave mode) has only one possible polarization.



   (a) Show that at low temperatures, the number of magnons per unit volume in a three-dimensional ferromagnet is given by

$$\frac{N_m}{V} = 2\pi \left(\frac{2m^* k_B T}{h^2}\right)^{3/2} \int_0^\infty \frac{\sqrt{x}}{e^x - 1}\, dx.$$

   The integral is given by $2.612\sqrt{\pi}/2$.

   (b) Use the result of part (a) to find an expression for the fractional reduction in magnetization, $(M(0) - M(T))/M(0)$. Write your answer in the form $(T/T_0)^{3/2}$, and estimate the constant T_0 for iron.

   (c) Calculate the heat capacity due to magnetic excitations in a ferromagnet at low temperature. You should find $C_V/Nk_B = (T/T_1)^{3/2}$, where T_1 differs from T_0 only by a numerical constant. Estimate T_1 for iron, and compare the magnon and phonon contributions to the heat capacity. (The Debye temperature of iron is 470 K.)

   (d) Consider a two-dimensional array of magnetic dipoles at low temperature. Assume that each elementary dipole can still point in any (three-dimensional) direction, so spin waves are still possible. Show that the integral for the total number of magnons diverges in this case. (This result is an indication that there can be no spontaneous magnetization in such a two-dimensional system. However, we will later consider a different two-dimensional model in which magnetization does occur.)

2. Black-body radiation.

   (a) Starting with $U = \sum_{\vec{k}} 2\frac{\hbar\omega(\vec{k})}{e^{\beta\hbar\omega(\vec{k})}-1}$, show that the total energy of the electromagnetic field per unit volume inside a cavity at tempterature T is

   $$\frac{U}{V} = \frac{(k_B T)^4}{\pi^2(\hbar c)^3}\int_0^\infty \frac{x^3}{e^x - 1}\,dx$$

   Begin by changing the sum to an integral, performing the angular integrals, then changing variables first from k to ω, then from ω to x.

   (b) Show that the total power per unit area escaping through a small hole in a wall of the cavity is $\frac{\pi^2}{15}\frac{c}{4}\frac{(k_B T)^4}{(\hbar c)^3}$
   Be sure to explain what you're doing at each step.

3. (Schroeder 7.66) Consider a collection of 10,000 atoms of rubidium-87 confined inside a box of volume $(10^{-5}m)^3$.

   (a) Calculate $\mathcal{E}_0$, the energy of the ground state ($n_x = 1, n_y = 1, n_z = 1$). (Express your answer in both joules and electron-volts.)

   (b) Calculate the condensation temperature, and compare $k_B T_c$ to $\mathcal{E}_0$.

   (c) Suppose that $T = 0.9T_c$. How many atoms are in the ground state? How close is the chemical potential to the ground-state energy? How many atoms are in each of the (threefold-degenerate) first excited states?

   (d) Repeat parts (b) and (c) for the case of 10^6 atoms, confined to the same volume. Discuss the conditions under which the number of atoms in the ground state will be much greater than the number in the first excited state.