Notes on a Few Topics in Special Relativity

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Given here are notes on a few topics in Special Relativity.

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The notes supplement material in Chapter 1 *Relativity* of the course text *Modern Physics* by Raymond A. Serway, Clement J. Moses and Curt A. Moyer, Saunders College Publishing, 2nd ed., (1997). All text equations referred to are given explicitly in these notes.

From an address by Hermann Minkowski

> The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth, space by itself, and time by itself, are destined to fade away into mere shadows, and only a kind of union of the two to remain an independent reality.

H. Minkowski, *Space and Time*, an address delivered at the 80th Assembly of German Natural Scientists and Physicians, Cologne, Germany, September 21, 1908.

Derivation of the Lorentz transformation equations

We derive the Lorentz transformation equations given by text Eqs. (1.25) to (1.28):

$$x' = \gamma(x - vt) \tag{1}$$

$$y' = y \tag{2}$$

$$z' = z \tag{3}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{4}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{5}$$

and

$$\beta = v/c \tag{6}$$

We derive the above using the two postulates of Special Relativity plus the assumption that space and time are homogeneous, that is, that all points in space and time are equivalent.

The S and S' inertial frames have been set up such that the x and x' axes coincide and the y and y' axes and z and z' axes are parallel. Also, seen from S, S' moves in the positive x-direction with speed v and, seen from S', S moves in the negative x'-direction with speed v. Furthermore, it is imagined that in each inertial frame there is an infinite set of recording clocks at rest in the frame and synchronized with each other. Clocks in both frames are set to zero when the origins O and O' coincide.

Let a physical event occur at point P. An observer attached to S specifies the location and time of the event by the four numbers x, y, z, t, that is, by the spacetime coordinates (x, y, z, t) of the event. The spacetime coordinates (x, y, z, t) of the event are recorded at the location (x, y, z) of the event at the time t it occurs. Measurements thus recorded are available to be picked up and analysed by an experimenter. The observer may be thought of as this experimenter. Similarly, an observer attached to S' specifies the location and time of the same event by the spacetime coordinates (x', y', z', t').

We wish to determine the functional relationships between spacetime coordinates in the two inertial frames. Firstly, the relationships must be linear, that is,

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t \tag{7}$$

$$y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t \tag{8}$$

$$z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t \tag{9}$$

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t \tag{10}$$

where the 16 quantities a_{ij} are real. If this were not the case, length and time intervals would depend upon where and when they were measured which is contrary to the assumption that space and time are homogeneous.

We now determine the 16 quantities a_{ij}. Eleven will be determined from the way the frames have been set up and from the assumption that space and time are homogeneous and five will be determined from the two postulates of Special Relativity.

We begin by looking at the way the two frames have been set up. Since the xz-frame is the same as the $x'z'$-frame (that is, $y = 0 \iff y' = 0$), it follows that $a_{21} = a_{23} = a_{24} = 0$, so $y' = a_{22}y$. Similarly, since the xy-frame is the same as the $x'y'$-frame (that is, $z = 0 \iff z' = 0$), it follows that $a_{31} = a_{32} = a_{34} = 0$, so $z' = a_{33}z$. Also, the y- and z-axes are perpendicular to the direction of motion of the frames but their directions are otherwise arbitrary. The assumption that space is homogeneous thus means that x' and t' cannot depend upon y and z, that is, $a_{12} = a_{13} = a_{41} = a_{43} = 0$. We have now determined that 10 of the a_{ij} are zero.

The point O' is specified by $x' = 0$ and $x = vt$ for all t, substitution of which into Eq. (**??**) yields

$$a_{14} = -a_{11}v \tag{11}$$

so Eqs. (**??**) to (**??**) have now become

$$x' = a_{11}(x - vt) \tag{12}$$

$$y' = a_{22}y \tag{13}$$

$$z' = a_{33}z \tag{14}$$

$$t' = a_{41}x + a_{44}t \tag{15}$$

The remaining 5 a_{ij} will be determined from the two postulates of Special Relativity. Einstein's first postulate (the Principle of Relativity) and the assumption that space is homogeneous require that $a_{22} = a_{33} = 1$, that is,

$$y' = y \tag{16}$$

$$z' = z \tag{17}$$

since, if this were not the case, lengths perpendicular to the direction of motion could be different in equivalent inertial frames, so there would be an asymmetry depending on the direction of motion, that is, depending on whether the motion is to the right or to the left. This is ruled out by the assumption that space is homogeneous.

We now use Einstein's second postulate (the constancy of the speed of light) to determine the 3 remaining a_{ij}. Suppose a burst of light begins spreading out in vacuum from the common origins of the two frames when the two origins coincide at $t' = t = 0$. An observer in S will measure that at a later time$t$, the light will have reached all points on a sphere surrounding O with radius ct:

$$x^2 + y^2 + z^2 = (ct)^2 \tag{18}$$

Similarly, according to the postulate of the constancy of the speed of light, an observer in S' will measure that at a later time t', the light will have reached all points on a sphere surrounding O' with radius ct':

$$x'^2 + y'^2 + z'^2 = (ct')^2 \tag{19}$$

Substitution of Eqs. (??), (??), (??) and (??) into Eq. (??) yields

$$a_{11}^2 - c^2 a_{41}^2 = 1 \tag{20}$$

$$v a_{11}^2 + c^2 a_{41} a_{44} = 0 \tag{21}$$

$$c^2 a_{44}^2 - v^2 a_{11}^2 = c^2 \tag{22}$$

whose solutions are

$$a_{11} = a_{44} = \gamma \tag{23}$$

$$a_{41} = -\beta\gamma/c^2 \tag{24}$$

Beta as a function of gamma

Eq. (??) gives γ in terms of β. In some problems, γ is given and β is needed. It follows from Eq. (??) that

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} \tag{25}$$

Lorentz boost in terms of rapidity

The Lorentz transformation equations Eqs. (??) to (??) (or Lorentz boost equations) are expressed in terms of the speed parameter β. They may also be expressed in terms of the boost paramater or rapidity ζ defined by

$$\zeta = \tanh^{-1}\beta \tag{26}$$

from which

$$\cosh\zeta = \gamma \tag{27}$$

and

$$\sinh\zeta = \beta\gamma \tag{28}$$

Accordingly, the first and fourth Lorentz transformation Eqs. (??) and (??) may be written as

$$x' = x\cosh\zeta - x_0 \sinh\zeta \tag{29}$$

$$x_0' = x_0\cosh\zeta - x\sinh\zeta \tag{30}$$

where

$$x_0 = ct \tag{31}$$

from which

$$x = x'\cosh\zeta + x_0'\sinh\zeta \tag{32}$$

$$x_0 = x_0'\cosh\zeta + x'\sinh\zeta \tag{33}$$

which are the inverse transformations given by text Eq. (1.30).

Four-vectors

We define coordinates x^0, x^1, x^2, x^3 by

$$(x^0, x^1, x^2, x^3) = (ct, x, y, z) \tag{34}$$

and coordinates x_0, x_1, x_2, x_3 by

$$(x_0, x_1, x_2, x_3) = (ct, -x, -y, -z) \tag{35}$$

Then Eqs. (??) and (??) can be expressed compactly as

$$x'^{\mu} x'_{\mu} = x^{\mu} x_{\mu} \tag{36}$$

where summation of the repeated Greek index over 0, 1, 2, 3 is implied. (This is Einstein's summation convention.)

x^0, x^1, x^2, x^3 and x_0, x_1, x_2, x_3 are components of the contravariant coordinate 4-vector and covariant coordinate 4-vector, respectively, and are related by

$$x^\mu = g^{\mu\nu} x_\nu \tag{37}$$

where the metric tensor $g^{\mu\nu}$ is defined as

$$g^{00} = -g^{11} = -g^{22} = -g^{33} = 1 \tag{38}$$

and

$$g^{\mu\nu} = 0 \quad \text{when} \quad \mu \neq \nu \tag{39}$$

Any quantities A^0, A^1, A^2, A^3 which transform under a Lorentz boost like x^0, x^1, x^2, x^3 are said to be components of a contravariant 4-vector. It follows that

$$A'^\mu A'_\mu = A^\mu A_\mu \tag{40}$$

that is, $A^\mu A_\mu$ is a Lorentz invariant. Similarly, if B^0, B^1, B^2, B^3 are components of a contravariant 4-vector, then $A^\mu B_\mu$ is a Lorentz invariant.

Matrix form of the Lorentz transformation equations

The Lorentz transformation Eqs. (**??**) to (**??**) can be written compactly as the matrix equation

$$x' = Lx \tag{41}$$

where

$$x' = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} \tag{42}$$

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{43}$$

and

$$L = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{44}$$

The Lorentz boost is specified by the real matrix $L = L(\zeta)$ which has the following properties:

$$\det L = 1 \tag{45}$$

$$\widetilde{L}(\zeta) = L(\zeta) \tag{46}$$

$$L^{-1}(\zeta) = L(-\zeta) \tag{47}$$

$$L(\zeta_1)L(\zeta_2) = L(\zeta_1 + \zeta_2) \tag{48}$$

where $\widetilde{L}$ and L^{-1} are the transpose and inverse of L, respectively. When written in terms of speeds rather than rapidities, Eq. (**??**) is

$$L(v_1)L(v_2) = L\left(\frac{v_1 + v_2}{1 + v_1 v_2/c^2}\right) \tag{49}$$

which expresses the relativistic addition of velocities.

The covariant coordinate 4-vector corresponds to the column matrix gx where

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{50}$$

Eq. (**??**) then is

$$\widetilde{x}' g x' = \widetilde{x} g x \tag{51}$$

which, using Eq. (**??**), yields

$$\widetilde{L} g L = g \tag{52}$$

from which

$$g \widetilde{L} g = L^{-1} \tag{53}$$

Eq. (**??**) is satisfied by every matrix L corresponding to a Lorentz boost.

Derivation of the relativistic expression for kinetic energy

The force $\mathbf{F}$ on a particle with rest mass m is the rate of change its momentum $\mathbf{p}$ as given by text Eq. (1.36):

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{54}$$

where, as given by text Eq. (1.35):

$$\mathbf{p} = \gamma m \mathbf{u} \tag{55}$$

where

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} \tag{56}$$

where $\mathbf{u}$ is the velocity of the particle.

Eq. (**??**) with $\mathbf{p}$ given by Eq. (**??**) is the relativistic generalization of Newton's Second Law.

The kinetic energy K of the particle is given by text Eq. (1.42):

$$K = (\gamma - 1) m c^2 \tag{57}$$

the nonrelativistic limit of which is

$$K = \frac{1}{2} m u^2 \tag{58}$$

We show that Eq. (**??**) follows from the definition

$$K = \int \mathbf{F} \cdot d\mathbf{r} \tag{59}$$

where the integration is over the path taken by the particle starting from rest. Using Eqs. (**??**) and (**??**) and $d\mathbf{r} = \mathbf{u} dt$ it follows that

$$K = m \int (\gamma \mathbf{u} \cdot d\mathbf{u} + u^2 d\gamma) \tag{60}$$

which, using

$$d\gamma = \gamma^3 \mathbf{u} \cdot d\mathbf{u} / c^2 \tag{61}$$

becomes

$$K = mc^2 \int_1^{\gamma} d\gamma = (\gamma - 1) m c^2 \tag{62}$$

Speed and rapidity as functions of momentum

Eq. (**??**) with Eq. (**??**) gives the momentum $\mathbf{p}$ of the particle as a function of its velocity $\mathbf{u}$. It is often useful to express u and γ in terms of p.

It follows from Eqs. (**??**) and (**??**) that

$$u = \frac{c}{\sqrt{1 + (mc/p)^2}} \tag{63}$$

and

$$\gamma = \sqrt{1 + (p/mc)^2} \tag{64}$$

It follows from Eq. (**??**) that a particle with rest mass $m = 0$ travels at the speed of light c.

It follows also that in the nonrelativitic limit,

$$u = p/m \tag{65}$$

Speed as a function of kinetic energy

Eq. (**??**) with Eq. (**??**) gives the kinetic energy K of a particle with rest mass m in terms of its speed u. It is often useful to express u in terms of K.

It follows from Eqs. (**??**) and (**??**) that

$$u = c\sqrt{1 - [1 + (K/mc^2)^{-2}]} \tag{66}$$

Momentum as a function of kinetic energy

Eq. (**??**) with Eq. (**??**) gives the kinetic energy K of a particle of rest mass m in terms of its speed u. It is often useful to express the momentum p of the particle in terms of K.

It follows from Eqs. (**??**) and (**??**) that

$$p = \sqrt{K(K + 2mc^2)}/c \tag{67}$$

Total energy E

The total energy E of a particle of rest mass m moving with velocity $\mathbf{u}$ is the sum of its kinetic energy K and rest mass energy mc^2:

$$E = K + mc^2 \tag{68}$$

It follows from Eq. (**??**) that

$$E = \gamma mc^2 \tag{69}$$

which is text Eq. (1.44). Eq. (**??**) expresses Einstein's famous equivalence of mass with energy.

Energy as a function of momentum

Eq. (**??**) with Eq. (**??**) gives the energy E of a particle with rest mass m as a function of its speed u. It is often useful to express E in terms of p.

It follows from Eqs. (**??**) and (**??**) that

$$E = \sqrt{p^2c^2 + m^2c^4} \tag{70}$$

It follows from Eq. (**??**) that the energy E and momentum p of a particle with rest mass $m = 0$ are related by

$$E = pc \tag{71}$$

Energy-momentum 4-vector

In frame S, a particle of rest mass m has velocity $\mathbf{u}$; its momentum $\mathbf{p}$ and energy E are given by text Eqs. (1.35) and (1.44):

$$\mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}} \tag{72}$$

$$E = \frac{mc^2}{\sqrt{1-u^2/c^2}}. \tag{73}$$

In frame S', which moves along the x-axis of S with speed v, the velocity of the particle is $\mathbf{u}'$ and its momentum $\mathbf{p}'$ and energy E' are

$$\mathbf{p}' = \frac{m\mathbf{u}'}{\sqrt{1-{u'}^2/c^2}} \tag{74}$$

$$E' = \frac{mc^2}{\sqrt{1-{u'}^2/c^2}}. \tag{75}$$

It follows from Eqs. (**??**) to (**??**) that

$${E'}^2 - (p'c)^2 = E^2 - (pc)^2 = m^2c^4. \tag{76}$$

The relationship between the momentum and energy in S and S' follows from the Lorentz velocity transformations given by text Eqs. (1.32) to (1.34):

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2} \tag{77}$$

$$u'_y = \frac{u_y}{\gamma(1 - u_x v/c^2)} \tag{78}$$

$$u'_z = \frac{u_z}{\gamma(1 - u_x v/c^2)} \tag{79}$$

where γ is given by Eq. (**??**). It follows that

$$\frac{1}{\sqrt{1-{u'}^2/c^2}} = \frac{\gamma(1-u_x v/c^2)}{\sqrt{1-u^2/c^2}} \tag{80}$$

so

$$p'_x = \gamma(p_x - \beta p_0) \tag{81}$$

$$p'_y = p_y \tag{82}$$

$$p'_z = p_z \tag{83}$$

$$p'_0 = \gamma(p_0 - \beta p_x) \tag{84}$$

where

$$p_0 = E/c. \tag{85}$$

Eqs. (**??**) and (**??**) may be written as

$$p'_x = p_x \cosh\zeta - p_0 \sinh\zeta \tag{86}$$

$$p'_0 = p_0 \cosh\zeta - p_x \sinh\zeta \tag{87}$$

where u is the rapidity Eq. (**??**) from which

$$p_x = p'_x \cosh\zeta + p'_0 \sinh\zeta \tag{88}$$

$$p_0 = p'_0 \cosh\zeta + p'_x \sinh\zeta \tag{89}$$

The quantities $(p^0, p^1, p^2, p^3) = (E/c, p_x, p_y, p_z)$ transform under Lorentz transformations like (ct, x, y, z) and are said to be components of the energy-momentum 4-vector.

It follows from Eqs. (**??**) and (**??**) and Eqs. (**??**) and (**??**) that

$${p'}^{\mu}{p'}_{\mu} = p^{\mu}p_{\mu} = (mc)^2 \tag{90}$$

Lorentz transformation of electric and magnetic fields

In frame S, a particle of rest mass m and charge q moves with velocity $\mathbf{u}$ in an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ and experiences a force

$$\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}) \tag{91}$$

that is,

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}) \tag{92}$$

where $\mathbf{p}$ is the momentunm of the particle in S. (We use Gaussian units for convenience here.) In frame S', which moves along the x-axis of S with speed v, the velocity of the particle is $\mathbf{u}'$ and the particle experiences a force

$$\mathbf{F}' = q(\mathbf{E}' + \frac{\mathbf{u}'}{c} \times \mathbf{B}') \tag{93}$$

where $\mathbf{E}'$ and $\mathbf{B}'$ are the electric and magnetic fields, respectively, in S', that is,

$$\frac{d\mathbf{p}'}{dt'} = q(\mathbf{E}' + \frac{\mathbf{u}'}{c} \times \mathbf{B}') \tag{94}$$

where $\mathbf{p}'$ is the momentum of the particle in S'.

Eqs. (**??**) and (**??**) are an example of Einstein's first postulate of the special theory of relativity (see text page 2): the laws of physics have the same mathematical form in inertial frames moving with constant velocity with respect to each other.

Eqs. (**??**) and (**??**) imply a relationship between the fields in S and S'. The left side of Eq. (**??**) can be written in terms of S frame quantities using Eqs. (**??**) to (**??**) and

$$\frac{dt'}{dt} = \gamma(1 - u_x v/c^2) \tag{95}$$

which follows from text Eq. (1.28), and

$$\frac{dp_0}{dt} = \frac{q}{c}\mathbf{u} \cdot \mathbf{E} \tag{96}$$

which follows from

$$dK = \mathbf{F} \cdot d\mathbf{r} \tag{97}$$

and Eq. (**??**). Finally, then

$$E'_x = E_x \tag{98}$$

$$E'_y = \gamma(E_y - \beta B_z) \tag{99}$$

$$E'_z = \gamma(E_z + \beta B_y) \tag{100}$$

$$B'_x = B_x \tag{101}$$

$$B'_y = \gamma(B_y + \beta E_z) \tag{102}$$

$$B'_z = \gamma(B_z - \beta E_y) \tag{103}$$

Eqs. (**??**) to (**??**) show that electric and magnetic fields have no independent existence: an electric or magnetic field in one frame will appear as a mixture of electric and magnetic fields in another frame. The fields are thus interrelated and one refers instead to the electromagnetic field in a given frame. This is formulated mathematically by construction of the second-rank antisymmetric electromagnetic field tensor

$$F = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{104}$$

which transforms under Lorentz transformations according to

$$F' = LFL \tag{105}$$

where L is given by Eq. (**??**).

Further information about the mathematical properties of the space-time of special relativity and the transformation of electromagnetic fields may be found in J.D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, Inc., 2nd ed., 1975, Chapter 11.