

# Solved Problems on Quantum Mechanics in One Dimension

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Given here are solutions to 15 problems on Quantum Mechanics in one dimension.

The solutions were used as a learning-tool for students in the introductory undergraduate course Physics 200 *Relativity and Quanta* given by Malcolm McMillan at UBC during the 1998 and 1999 Winter Sessions. The solutions were prepared in collaboration with Charles Asman and Adam Monahan who were graduate students in the Department of Physics at the time.

The problems are from Chapter 5 *Quantum Mechanics in One Dimension* of the course text *Modern Physics* by Raymond A. Serway, Clement J. Moses and Curt A. Moyer, Saunders College Publishing, 2nd ed., (1997).

## Planck's Constant and the Speed of Light.

When solving numerical problems in Quantum Mechanics it is useful to note that the product of Planck's constant

$$h = 6.6261 \times 10^{-34} \text{ J s} \quad (1)$$

and the speed of light

$$c = 2.9979 \times 10^8 \text{ m s}^{-1} \quad (2)$$

is

$$hc = 1239.8 \text{ eV nm} = 1239.8 \text{ keV pm} = 1239.8 \text{ MeV fm} \quad (3)$$

where

$$\text{eV} = 1.6022 \times 10^{-19} \text{ J} \quad (4)$$

Also,

$$\hbar c = 197.32 \text{ eV nm} = 197.32 \text{ keV pm} = 197.32 \text{ MeV fm} \quad (5)$$

where  $\hbar = h/2\pi$ .

## Wave Function for a Free Particle

Problem 5.3, page 224

A free electron has wave function

$$\Psi(x, t) = \sin(kx - \omega t) \quad (6)$$

- Determine the electron's de Broglie wavelength, momentum, kinetic energy and speed when  $k = 50 \text{ nm}^{-1}$ .
- Determine the electron's de Broglie wavelength, momentum, total energy, kinetic energy and speed when  $k = 50 \text{ pm}^{-1}$ .

### Solution

The equations relating the speed  $v$ , momentum  $p$ , de Broglie wavelength  $\lambda$ , wave number  $k$ , kinetic energy  $E$ , angular frequency  $\omega$  and group velocity  $v_g$  for a nonrelativistic particle of mass  $m$  are:

$$p = mv = \frac{h}{\lambda} = \hbar k \quad (7)$$

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} = \hbar\omega \quad (8)$$

$$v_g = \frac{d\omega}{dk} = v \quad (9)$$

When  $k = 50 \text{ nm}^{-1}$ ,

$$\lambda = 126 \text{ pm} \quad p = 9.87 \text{ keV}/c \quad (10)$$

and, for an electron ( $m = 511 \text{ keV}/c^2$ ),

$$E = 95.2 \text{ eV} \quad v = 1.93 \times 10^{-2}c \quad (11)$$

The equations relating the speed  $v$ , momentum  $p$ , de Broglie wavelength  $\lambda$ , wave number  $k$ , total energy  $E$ , kinetic energy  $K$ , angular frequency  $\omega$  and group velocity  $v_g$  for a relativistic particle of mass  $m$  are:

$$p = \gamma mv = \frac{h}{\lambda} = \hbar k \quad (12)$$

$$E = \gamma mc^2 = mc^2 + K = \sqrt{p^2 c^2 + m^2 c^4} = \hbar\omega \quad (13)$$

$$v_g = \frac{d\omega}{dk} = v = \frac{pc^2}{E} \quad (14)$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad (15)$$

$$\beta = v/c \quad (16)$$

When  $k = 50 \text{ pm}^{-1}$ ,

$$\lambda = 126 \text{ fm} \quad p = 9.87 \text{ MeV}/c \quad (17)$$

and, for an electron ( $m = 511 \text{ keV}/c^2$ ),

$$E = 9.88 \text{ MeV} \quad K = 9.37 \text{ MeV} \quad v = 0.9987c \quad (18)$$

### Potential Energy of a Particle

Problem 5.5, page 224

In a region of space, a particle with mass  $m$  and with zero energy has a time-independent wave function

$$\psi(x) = Axe^{-x^2/L^2} \quad (19)$$

where  $A$  and  $L$  are constants.

- Determine the potential energy  $U(x)$  of the particle.

### Solution

Text Eq. (5.13) is the time-independent Schrödinger equation for the wavefunction  $\psi(x)$  of a particle of mass  $m$  in a potential  $U(x)$ :

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x) \quad (20)$$

When a particle with zero energy has wavefunction  $\psi(x)$  given by Eq. (19), it follows on substitution into Eq. (20) that

$$U(x) = \frac{2\hbar^2}{mL^4} \left( x^2 - \frac{3L^2}{2} \right). \quad (21)$$

$U(x)$  is a parabola centred at  $x = 0$  with  $U(0) = -3\hbar^2/mL^2$ .

### Photon Energy From a Transition in an Infinite Square Well Potential

Problem 5.9, page 224

A proton is confined in an infinite square well of width 10 fm. (The nuclear potential that binds protons and neutrons in the nucleus of an atom is often approximated by an infinite square well potential.)

- Calculate the energy and wavelength of the photon emitted when the proton undergoes a transition from the first excited state ( $n = 2$ ) to the ground state ( $n = 1$ ).
- In what region of the electromagnetic spectrum does this wavelength belong?

#### Solution

Text Eq. (5.17) gives the energy  $E_n$  of a particle of mass  $m$  in the  $n$ th energy state of an infinite square well potential with width  $L$ :

$$E_n = \frac{n^2\hbar^2}{8mL^2} \quad (22)$$

The energy  $E$  and wavelength  $\lambda$  of a photon emitted as the particle makes a transition from the  $n = 2$  state to the  $n = 1$  state are

$$E = E_2 - E_1 = \frac{3\hbar^2}{8mL^2} \quad (23)$$

$$\lambda = \frac{hc}{E}. \quad (24)$$

For a proton ( $m=938 \text{ MeV}/c^2$ ),  $E = 6.15 \text{ MeV}$  and  $\lambda = 202 \text{ fm}$ . The wavelength is in the gamma ray region of the spectrum.

### Wave Functions for a Particle in an Infinite Square Well Potential

Problem 5.11, page 225

A particle with mass  $m$  is in an infinite square well potential with walls at  $x = -L/2$  and  $x = L/2$ .

- Write the wave functions for the states  $n = 1$ ,  $n = 2$  and  $n = 3$ .

#### Solution

Text Eqs. (5.18) and (5.19) give the normalized wave functions for a particle in an infinite square well potential with walls at  $x = 0$  and  $x = L$ . To obtain the wavefunctions  $\psi_n(x)$  for a particle in an infinite square potential with walls at  $x = -L/2$  and  $x = L/2$  we replace  $x$  in text Eq. (5.18) by  $x + L/2$ :

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi(x + L/2)}{L}\right) \quad (25)$$

which satisfies  $\psi_n(-L/2) = \psi_n(L/2) = 0$  as required. Thus,

$$\psi_1(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right) \quad (26)$$

$$\psi_2(x) = -\sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right) \quad (27)$$

$$\psi_3(x) = -\sqrt{\frac{2}{L}} \cos\left(\frac{3\pi x}{L}\right) \quad (28)$$

### Position Probability for a Particle in an Infinite Square Well Potential

Problem 5.16, page 225

A particle is in the  $n$ th energy state  $\psi_n(x)$  of an infinite square well potential with width  $L$ .

- Determine the probability  $P_n(1/a)$  that the particle is confined to the first  $1/a$  of the width of the well.
- Comment on the  $n$ -dependence of  $P_n(1/a)$ .

#### Solution

The wave function  $\psi_n(x)$  for a particle in the  $n$ th energy state in an infinite square box with walls at  $x = 0$  and  $x = L$  is

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right). \quad (29)$$

The probability  $P_n(1/a)$  that the electron is between  $x = 0$  and  $x = L/a$  in the state  $\psi_n(x)$  is

$$P_n\left(\frac{1}{a}\right) = \int_0^{L/a} |\psi_n(x)|^2 dx = \frac{2}{L} \int_0^{L/a} \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{1}{a} - \frac{\sin(2n\pi/a)}{2n\pi} \quad (30)$$

$P_n(1/a)$  is the probability that the particle in the state  $\psi_n(x)$  is confined to the first  $1/a$  of the width of the well. The sinusoidal  $n$ -dependent term decreases as  $n$  increases and vanishes in the limit of large  $n$ :

$$P_n\left(\frac{1}{a}\right) \rightarrow \frac{1}{a} \text{ as } n \rightarrow \infty \quad (31)$$

$P_n(1/a) = 1/a$  is the classical result. The above analysis is consistent with the correspondence principle, which may be stated symbolically as

$$\text{quantum physics} \rightarrow \text{classical physics as } n \rightarrow \infty \quad (32)$$

where  $n$  is a typical quantum number of the system.

### Application of Quantum Mechanics to a Macroscopic Object

Problem 5.19, page 225

A 1.00 g marble is constrained to roll inside a tube of length  $L = 1.00$  cm. The tube is capped at both ends.

- Modelling this as a one-dimensional infinite square well, determine the value of the quantum number  $n$  if the marble is initially given an energy of 1.00 mJ.
- Calculate the excitation energy required to promote the marble to the next available energy state.

#### Solution

The allowed energy values  $E_n$  for a particle of mass  $m$  in a one-dimensional infinite square well potential of width  $L$  are given by Eq. (22) from which

$$n = 4.27 \times 10^{28} \quad (33)$$

when  $E_n = 1.00$  mJ.

The excitation energy  $E$  required to promote the marble to the next available energy state is

$$E = E_{n+1} - E_n = \frac{(2n+1)h^2}{8mL^2} = 4.69 \times 10^{-32} \text{ J.} \quad (34)$$

This example illustrates the large quantum numbers and small energy differences associated with the behavior of macroscopic objects.

### Energy Levels for a Particle in a Finite Square Well Potential

Problem 5.20, page 225

A particle with energy  $E$  is bound in a finite square well potential with height  $U$  and width  $2L$  situated at  $-L \leq x \leq +L$ .

The potential is symmetric about the midpoint of the well. The stationary state wave functions are either symmetric or antisymmetric about this point.

• Show that for  $E < U$ , the conditions for smooth joining of the interior and exterior wave functions leads to the following equation for the allowed energies of the symmetric wave functions:

$$k \tan kL = \alpha \quad (35)$$

where

$$\alpha = \sqrt{\frac{2m(U-E)}{\hbar^2}}. \quad (36)$$

and

$$k = \sqrt{\frac{2mE}{\hbar^2}} \quad (37)$$

$k$  is the wave number of oscillation in the interior of the well.

• Show that Eq. (35) can be rewritten as

$$k \sec kL = \frac{\sqrt{2mU}}{\hbar} \quad (38)$$

• Apply this result to an electron trapped at a defect site in a crystal, modeling the defect as a finite square well potential with height 5 eV and width 200 pm.

### Solution

The wavefunction  $\psi(x)$  for a particle with energy  $E$  in a potential  $U(x)$  satisfies the time-independent Schrödinger equation Eq. (20).

Inside the well ( $-L \leq x \leq L$ ), the particle is free. The wavefunction symmetric about  $x = 0$  is

$$\psi(x) = A \cos kx \quad \text{where } k = \sqrt{\frac{2mE}{\hbar^2}}. \quad (39)$$

Outside the well ( $-\infty < x < -L$  and  $L < x < \infty$ ), the potential has constant value  $U > E$ . The wavefunction symmetric about  $x = 0$  is

$$\psi(x) = B e^{-\alpha|x|} \quad \text{where } \alpha = \sqrt{\frac{2m(U-E)}{\hbar^2}}. \quad (40)$$

$\psi(x)$  and its derivative are continuous at  $x = L$ :

$$A \cos kL = B e^{-\alpha L} \quad (41)$$

$$Ak \sin kL = B \alpha e^{-\alpha L} \quad (42)$$

from which

$$k \tan kL = \alpha \quad (43)$$

or, alternatively,

$$\theta \sec \theta = \pm a \quad (44)$$

where

$$\theta = kL \quad (45)$$

and

$$a = \sqrt{\frac{2mUL^2}{\hbar^2}} \quad (46)$$

Eq. (44) are equations for the allowed values of  $k$ . The equation with the positive sign yields values of  $\theta$  in the first quadrant. The equation with the negative sign yields values of  $\theta$  in the third quadrant.

Solving Eq. (44) numerically for an electron in a well with  $U = 5$  eV and  $L = 100$  pm yields the ground state energy  $E = 2.43$  eV.

### Energy Levels for a Particle in a Semi-Infinite Square Well Potential

Problem 5.23, page 226

Consider a square well having an infinite wall at  $x = 0$  and a wall of height  $U$  at  $x = L$ .

- For the case  $E < U$ , obtain solutions to the Schrödinger equation Eq. (20) inside the well ( $0 \leq x \leq L$ ) and in the region beyond ( $x > L$ ) that satisfy the appropriate boundary conditions at  $x = 0$  and at  $x = \infty$ .
- Enforce the proper matching conditions at  $x = L$  to find an equation for the allowed energies of the system.
- Are there conditions for which no solution is possible? Explain.

#### Solution

The wavefunction  $\psi(x)$  for a particle with energy  $E$  in a potential  $U(x)$  satisfies the Schrödinger Eq. (20). Inside the well ( $0 \leq x \leq L$ ), the particle is free. The wavefunction is

$$\psi(x) = A \sin kx \quad \text{where } k = \sqrt{\frac{2mE}{\hbar^2}}. \quad (47)$$

Outside the well ( $L < x < \infty$ ), the potential has constant value  $U > E$ . The wavefunction is

$$\psi(x) = B e^{-\alpha x} \quad \text{where } \alpha = \sqrt{\frac{2m(U - E)}{\hbar^2}}. \quad (48)$$

$\psi(x)$  and its derivative are continuous at  $x = L$ :

$$A \sin kL = B e^{-\alpha L} \quad (49)$$

$$Ak \cos kL = -B \alpha e^{-\alpha L} \quad (50)$$

from which

$$k \cot kL = -\alpha \quad (51)$$

or, alternatively,

$$\theta \csc \theta = \pm a \quad (52)$$

where  $\theta$  and  $a$  are given by Eqs. (45) and (46).

Eq. (52) are equations for the allowed values of  $k$ . The equation with the positive sign yields values of  $\theta$  in the second quadrant. The equation with the negative sign yields values of  $\theta$  in the fourth quadrant.

Since  $\sin \theta \leq \theta \quad \forall \theta$ , it follows from Eq. (52) that there are no bound states if  $UL^2 \leq \hbar^2/2m$ .

### Wave Function for the First Excited State of a Harmonic Oscillator

Problem 5.25, page 226

The wave function

$$\psi(x) = A x e^{-\alpha x^2} \quad (53)$$

describes a state of a harmonic oscillator provided the constant  $\alpha$  is chosen appropriately.

- Using the Schrödinger Eq. (20), determine an expression for  $\alpha$  in terms of the oscillator mass  $m$  and the classical frequency of vibration  $\omega$ .

- Determine the energy of this state and normalize the wave function.

#### Solution

Eq. (20) is the Schrödinger equation for a harmonic oscillator when

$$U(x) = \frac{1}{2} K x^2 = \frac{1}{2} m \omega^2 x^2 \quad \text{where} \quad \omega = \omega_{\text{classical}} = \sqrt{\frac{K}{m}} \quad (54)$$

$\psi(x)$  given by Eq. (53) satisfies Eq. (20) when

$$\alpha = \frac{m\omega}{2\hbar} \quad \text{and} \quad E = \frac{3}{2} \hbar \omega \quad (55)$$

Eq. (53) gives the wave function of the first excited state of the harmonic oscillator. Requiring that

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1 \quad (56)$$

yields

$$|A| = \left( \frac{32\alpha^3}{\pi} \right)^{1/4}. \quad (57)$$

### Average Position of a Particle

Problem 5.30, page 226

An electron is described by the wave function

$$\psi(x) = \begin{cases} 0 & \text{for } x < 0 \\ C e^{-x} (1 - e^{-x}) & \text{for } x > 0, \end{cases} \quad (58)$$

where  $x$  is in nm and  $C$  is a constant.

- Determine the value of  $C$  that normalizes  $\psi(x)$ .

- Where is the electron most likely to be found? That is, for what value of  $x$  is the probability of finding the electron the largest?

- Calculate the average position  $\langle x \rangle$  for the electron. Compare this result with the most likely position, and comment on the difference.

#### Solution

An electron is described by the wave function  $\psi(x)$  given by Eq. (58). Requiring that Eq. (56) holds yields

$$|C| = 2\sqrt{3} \text{ nm}^{-1/2}. \quad (59)$$

The most likely place  $x_m$  for the electron to be is where  $|\psi(x)|^2$  is maximum or, in this case, where  $\psi(x)$  is maximum. It follows from Eq. (58) that

$$x_m = \ln 2 \text{ nm} = 0.693 \text{ nm} \quad (60)$$

It follows from text Eqs. (5.11) and (5.31) that the average position  $\langle x \rangle$  of a particle in a stationary state  $\psi(x)$  is

$$\langle x \rangle = \int_{-\infty}^{+\infty} x |\psi(x)|^2 dx \quad (61)$$

It follows from Eqs. (58) and (61) that

$$\langle x \rangle = \frac{13}{12} \text{ nm} \simeq 1.083 \text{ nm}. \quad (62)$$

$\langle x \rangle > x_m$  because, according to Eq. (58), values of  $x > x_m$  are weighted more heavily in determining  $\langle x \rangle$ .

### Position Properties of a Quantum Oscillator in its Ground State

Problem 5.33, page 227

- Calculate  $\langle x \rangle$ ,  $\langle x^2 \rangle$  and  $\Delta x$  for a quantum oscillator in its ground state.

#### Solution

The normalized wave function  $\psi_0(x)$  for the ground state of the quantum oscillator is

$$\psi_0(x) = C_0 e^{-\alpha x^2} \quad (63)$$

where

$$C_0 = \left( \frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} \quad \alpha = \frac{m\omega}{2\hbar} \quad E = \frac{1}{2}\hbar\omega \quad (64)$$

$\psi_0(x)$  is the solution of the Schrödinger Eq. (20) when  $U(x)$  is the harmonic oscillator potential given by Eq. (54).

It follows from Eq. (61) that

$$\langle x \rangle = 0. \quad (65)$$

The uncertainty  $\Delta x$  in the position of a particle in a stationary state  $\psi(x)$  as

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \quad (66)$$

where

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 |\psi(x)|^2 dx. \quad (67)$$

For the ground state of the quantum oscillator,

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}}. \quad (68)$$

### Momentum Properties of a Quantum Oscillator in its Ground State



- Calculate  $\langle p \rangle$ ,  $\langle p^2 \rangle$  and  $\Delta p$  for a quantum oscillator in its ground state.

**Solution**

The average momentum  $\langle p \rangle$  of a particle in a stationary state  $\psi(x)$  is

$$\langle p \rangle = -i\hbar \int_{-\infty}^{+\infty} \psi^*(x) \frac{d\psi(x)}{dx} dx \quad (69)$$

The uncertainty  $\Delta p$  in the momentum of a particle in a stationary state  $\psi(x)$  is

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \quad (70)$$

where

$$\langle p^2 \rangle = -\hbar^2 \int_{-\infty}^{+\infty} \psi^*(x) \frac{d^2\psi(x)}{dx^2} dx \quad (71)$$

Eq. (63) gives the wave function of the ground state of the quantum oscillator. For this state,

$$\langle p \rangle = 0 \quad (72)$$

$$\Delta p = \sqrt{\frac{m\hbar\omega}{2}}. \quad (73)$$

**The Ground State of a Quantum Oscillator is an Optimum State of Position and Momentum**

$\psi(x)$  is said to be an optimum state of position and momentum if for this state

$$\Delta x \Delta p = \hbar/2 \quad (74)$$

That is, for an optimum state, the product of the uncertainties of position and momentum is the smallest value allowed by Heisenberg's Uncertainty Relation:

$$\Delta x \Delta p \geq \hbar/2 \quad (75)$$

- Show that the ground state of the harmonic oscillator is an optimum state.

**Solution**

It follows from Eqs. (68) and (73) that Eq. (74) holds for  $\psi(x)$  given by Eq. (63). That is, the ground state of the quantum oscillator is an optimum state of position and momentum.

**Average Energy and Uncertainty of Energy of a Particle in a Nonstationary State**

A particle in a potential well  $U(x)$  is initially in a state whose wavefunction  $\Psi(x, 0)$  is an equal-weight superposition of the ground state and first excited state wavefunctions:

$$\Psi(x, 0) = C[\psi_1(x) + \psi_2(x)] \quad (76)$$

- Show that the value  $C = 1/\sqrt{2}$  normalizes  $\Psi(x, 0)$ , assuming that  $\psi_1$  and  $\psi_2$  are themselves normalized.
- Determine  $\Psi(x, t)$  at any later time  $t$ .
- Show that the average energy  $\langle E \rangle$  for  $\Psi(x, t)$  is the arithmetic mean of the ground and first excited state energies  $E_1$  and  $E_2$ , that is  $\langle E \rangle = (E_1 + E_2)/2$ .
- Determine the uncertainty  $\Delta E$  of energy for  $\Psi(x, t)$ .

### Solution

The stationary state wavefunctions  $\psi_n(x)$  are solutions of Eq. (20) and are orthonormal:

$$\int_{-\infty}^{+\infty} \psi_m^*(x)\psi_n(x)dx = \delta_{mn} \quad (77)$$

where  $\delta_{mn}$  is the Kronecker  $\delta$ -function ( $\delta_{mm} = 1$ ;  $\delta_{mn} = 0$  when  $m \neq n$ ) and  $\psi_n(x)$  corresponds to energy  $E_n$ .

The wavefunction  $\Psi(x, t)$  is normalized:

$$\int_{-\infty}^{+\infty} |\Psi(x, t)|^2 dx = 1 \quad (78)$$

so

$$C = \frac{1}{\sqrt{2}}. \quad (79)$$

At time  $t$ ,

$$\Psi(x, t) = \frac{1}{\sqrt{2}} \left[ \psi_1(x)e^{-iE_1t/\hbar} + \psi_2(x)e^{-iE_2t/\hbar} \right]. \quad (80)$$

It follows from text Eqs. (5.37) and (5.40) that the average energy  $\langle E \rangle$  of the particle is

$$\langle E \rangle = i\hbar \int_{-\infty}^{+\infty} \Psi^*(x, t) \frac{\partial \Psi(x, t)}{\partial t} dx \quad (81)$$

and from text Eq. (5.41) that the uncertainty  $\Delta E$  in the energy is

$$\Delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2} \quad (82)$$

where

$$\langle E^2 \rangle = -\hbar^2 \int_{-\infty}^{+\infty} \Psi^*(x, t) \frac{\partial^2 \Psi(x, t)}{\partial t^2} dx \quad (83)$$

It follows from Eq. (80) that

$$\langle E \rangle = \frac{1}{2}(E_1 + E_2) \quad (84)$$

$$\Delta E = \frac{1}{2}(E_2 - E_1). \quad (85)$$

That is, the average energy  $\langle E \rangle$  for  $\Psi(x, t)$  is the arithmetic mean of the ground and first excited state energies  $E_1$  and  $E_2$ . Now,

$$\Delta E = E_2 - \langle E \rangle = \langle E \rangle - E_1 \quad (86)$$

so the uncertainty  $\Delta E$  in the energy is the difference between first excited state energy and the average energy, or, equivalently, between the average energy and the ground state energy.

### Average Position of a Particle in a Nonstationary State

Problem 5.40, page 227

•Determine the average position  $\langle x(t) \rangle$  of a particle with nonstationary state wave function  $\Psi(x, t)$  given by Eq. (80).

### Solution

Text Eq. (5.31) gives the average position  $\langle x(t) \rangle$  of a particle in a state  $\Psi(x, t)$  as

$$\langle x(t) \rangle = \int_{-\infty}^{+\infty} x |\Psi(x, t)|^2 dx \quad (87)$$

For a particle in the state  $\Psi(x, t)$  given by Eq. (80) it follows that

$$\langle x(t) \rangle = x_0 + a \cos(t/\tau) \quad (88)$$

where

$$x_0 = \frac{1}{2} \int_{-\infty}^{+\infty} x [\psi_1^2(x) + \psi_2^2(x)] dx \quad (89)$$

$$a = \int_{-\infty}^{+\infty} x \psi_1(x) \psi_2(x) dx \quad (90)$$

$$\tau = \frac{\hbar}{2\Delta E} \quad (91)$$

where  $\Delta E$  is given by Eq. (85) and where we have chosen the stationary state wave functions to be real.

The average position of the particle oscillates about  $x_0$  with amplitude  $a$  and period  $2\pi\tau$ .

It follows from the energy-time uncertainty relation [text Eq. (4.34)]:

$$\Delta E \Delta t \geq \hbar/2 \quad (92)$$

that  $\tau$  is the minimum time required to observe a change in the average position of the particle.

For an electron in an infinite square well potential with width  $L = 100$  pm,  $\tau = 5.83$  as.

For a proton in an infinite square well potential with width  $L = 10$  fm,  $\tau = 107$  as.