A holographic proof of the (1st order) coarse-grained GSL*

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Work to appear with Will Bunting (Caltech), Zicao Fu (Tsinghua U.), and Aron Wall.

*= + possible bonus track

The GSL

- Bekenstein (1973): $S_{\text{gen}} = \frac{A}{4G_N} + S_{\text{out}}$ is a non-decreasing function of time. (Assume Einstein-Hilbert gravity for this talk.)

- S_{out} coarse-grained or fine-grained?

- Wall (2011) proved the (1^{st} order perturbative in G_N) fine-grained GSL for super-renormalizeable field theories.

This talk derives a (1^{st} order perturbative) GSL for a *coarse-grained* S_{out} in arbitrary holographic theories.
Perturbative in G_N, so start with CFT on a fixed BH background and compute back-reaction later.

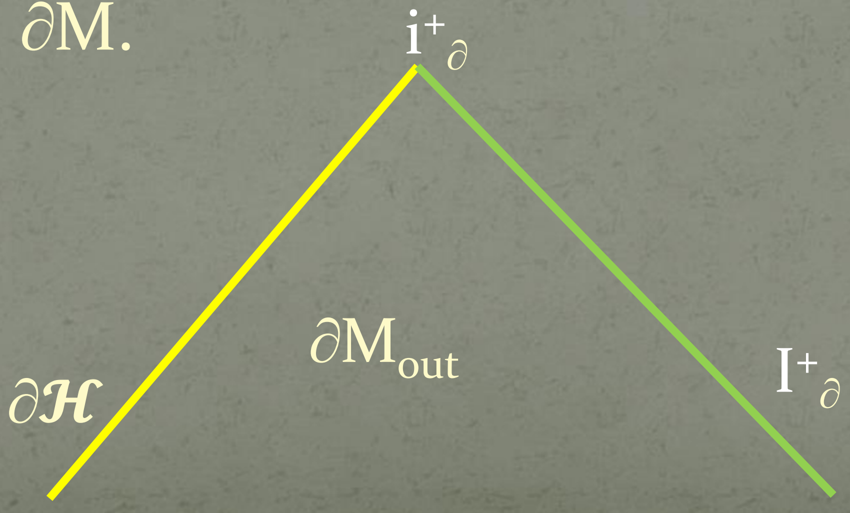
Interesting points:

1. It's really easy.
2. Applies to a new class of theories. Appears to need slightly stronger assumptions than Wall (2011).
3. A corollary is that the field theory on a fixed (non-dynamical) black hole background has non-increasing free energy. The bulk dual is that black holes whose horizons reach the boundary also have non-increasing F. A "Hawking area theorem" for black holes with non-compact horizons! (Droplets/funnels)
4. Our coarse-grained S_{out} at each time is just the (renormalized) area of some cut of the bulk event horizon. Essentially Causal Holographic Info.
   What is this in the CFT? (Kelly-Wall?)

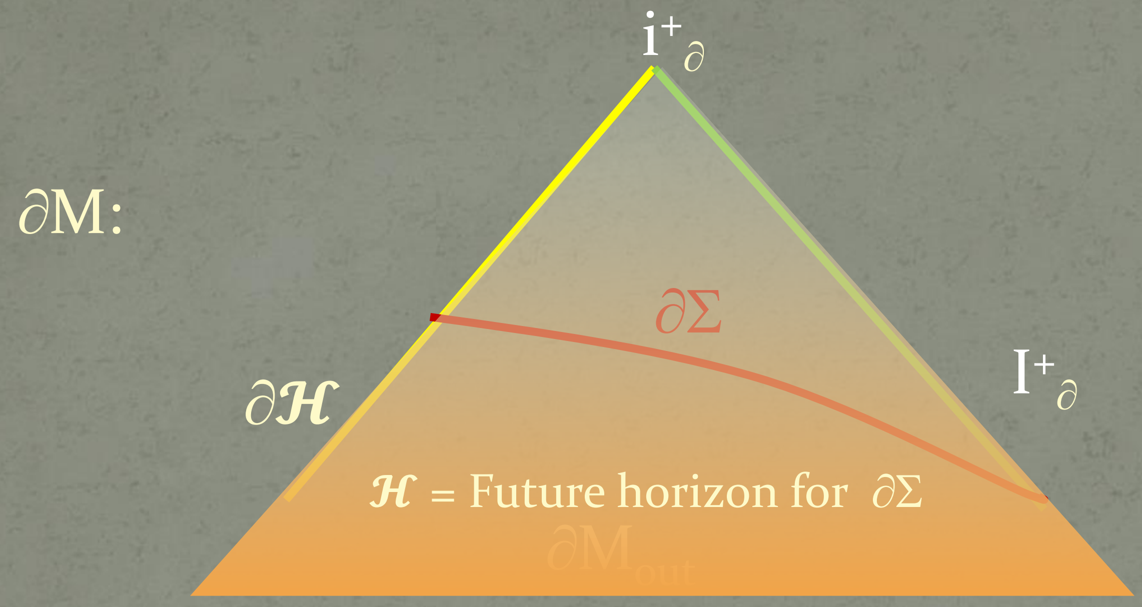
Cast of characters

- An asymptotically (locally) AdS spacetime M.
- It's boundary ∂M (w/ *fixed* boundary metric).
- A Killing horizon $\partial\mathcal{H}$ in ∂M and its endpoint i^+_∂.
- The region ∂M_{out} of ∂M outside $\partial\mathcal{H}$. An Asympt Flat ∂M_{out} is drawn below but not required.
- The bulk event horizon $\mathcal{H}$ defined by ∂M_{out}.
  Note that $\mathcal{H}$ is the past light cone of i^+_∂.
  Also $\partial\mathcal{H} = \mathcal{H} \cap \partial M$.

∂M:



Consider an achronal surface $\partial\Sigma$ in ∂M_{out} that ends on $\partial\mathcal{H}$.



This $\partial\Sigma$ defines a bulk causal wedge whose bifurcation surface B is a cut of $\mathcal{H}$ with boundary $\partial B = \partial\mathcal{H} \cap \partial\Sigma$ (up to issues at I^+_∂).

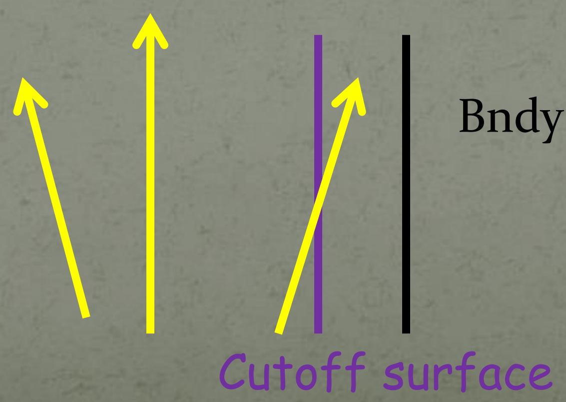
B has more area than the extremal surface with boundary ∂B. [Hubeny & Rangamani, Wall]
So B is naturally interpreted as a coarse-grained entropy
[e.g., as in Kelly & Wall].

Sketch of proof

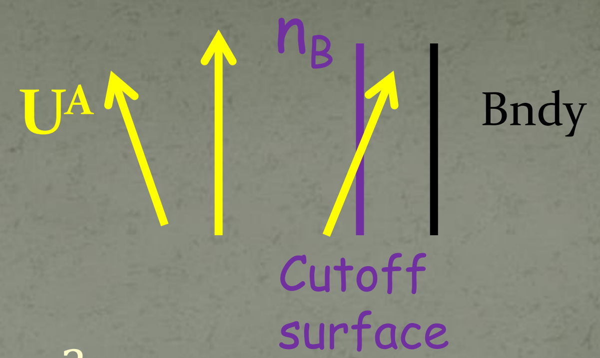
- The expansion of $\mathcal{H}$ is non-negative on each generator.

- $A_{ren} = A - A_{ct}$ can decrease when generators flow to infinity along the non-compact horizon. But a computation relates this area-flux the flux of the bndy stress tensor T_{ab} across $\partial\mathcal{H}$. Note: $\mathcal{H}$ becomes a Killing horizon near the bndy; gives control over divergent terms in A.

- One then allows the boundary metric to react to the flux of T_{ab} . Raychaudhuri's equation gives the change ΔA for the bndy black hole (as in the physical process 1^{st} law). This precisely cancels the CFT entropy decrease associated with the bulk area-flux to infinity. So ΔS_{gen} comes only from the (non-negative) local expansion in the bulk.



Framework



Bulk metric:

$$ds^2{}_{d+1} = G^{AB}\ dx^A dx^B = \left(\frac{l}{z}\right)^2 (dz^2 + g_{\alpha\beta}(z)dx^\alpha dx^\beta)$$

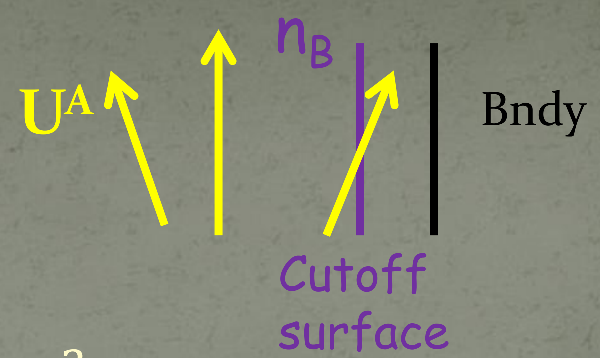
$$g_{\alpha\beta}(z) = \underbrace{g^{(0)}{}_{\alpha\beta} + z^2 g^{(2)}{}_{\alpha\beta} + \ldots + z^d\ \bar{g}^{(d)}{}_{\alpha\beta}}_{\text{Background } \bar{g}_{\alpha\beta}(z),\ \text{w/ corresp } \bar{G}_{AB}} + \underbrace{\mathrm{z}^{\mathrm{d}}\left(\frac{16\pi G_{d+1}}{d l^{d-1}}\right) T_{\alpha\beta} + \ldots}_{\text{Perturbation } \delta g_{\alpha\beta}(z),\ \text{w/ corresp } \delta G_{AB}}$$

Affinely parametrized generators $U^A = \bar{U}^A + \delta U^A$;

inward-pointing normal to cutoff surface $n_B = \bar{n}_B = \frac{l}{z}\delta_{Bz}$

Area flux $\propto n_A U^A$ vanishes in background ($\bar{U}^z = 0$)

$\boxed{\text{Easier for Ricci-flat } g^{(0)}_{\alpha\beta}.}$



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$$ds^2{}_{d+1} = G^{AB} \quad dx^A dx^B = \left(\frac{l}{z}\right)^2 (dz^2 + g_{\alpha\beta}(z) dx^\alpha dx^\beta)$$

$$g_{\alpha\beta}(z) = \underbrace{g^{(0)}{}_{\alpha\beta} \cancel{+z^2 g^{(2)}{}_{\alpha\beta} + \ldots + z^d \; \bar{g}^{(d)}{}_{\alpha\beta}}}_{\text{Background } \bar{g}_{\alpha\beta}(z),\ \text{w/ corresp } \bar{G}_{AB}} + \underbrace{\mathrm{z}^{\mathrm{d}} (\frac{16\pi G_{d+1}}{d l^{d-1}}) \, T_{\alpha\beta} + \ldots}_{\text{Perturbation } \delta g_{\alpha\beta}(z),\ \text{w/ corresp } \delta G_{AB}}$$

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The computation (∇n part)

$$\bar{n}_B = \frac{l}{z}\delta_{Bz}$$

Compute $\frac{d}{d\lambda}(n_A U^A) := \bar{U}^B \bar{\nabla}_B (n_A U^A) = \bar{U}^B \bar{\nabla}_B (\bar{n}_A \delta U^A)$

$$\bar{U}^B \bar{\nabla}_B \bar{n}_A = \frac{l}{z} \bar{U}^B \bar{\Gamma}^z_{AB} = - l z^{-2} \bar{U}^B g^{(0)}_{BA} = -l^{-1} \bar{U}^B \bar{G}_{AB}$$

$$\delta U^A \bar{U}^B \bar{\nabla}_B \bar{n}_A = -l^{-1} \bar{U}^B \bar{G}_{AB}\, \delta U^A$$

But U^A remains null after adding the perturbation:

$$\delta U^2 = 2\bar{U}^B \bar{G}_{AB}\, \delta U^A + \bar{U}^A \bar{U}^B \delta G_{AB}$$

So,

$$\delta U^A \bar{U}^B \bar{\nabla}_B \bar{n}_A = \frac{1}{2l} \bar{U}^A \bar{U}^B \delta G_{AB} = \frac{1}{2l} \bar{U}^\alpha \bar{U}^\beta l^2 z^{d-2} \frac{16\pi G_{d+1}}{d l^{d-1}} T_{\alpha\beta}$$

$$= \left(\frac{z}{l}\right)^{d-2} \frac{8\pi G_{d+1}}{d} T_{\alpha\beta} \bar{U}^\alpha \bar{U}^\beta + \ldots \quad (1)$$

The computation (∇δU part)

$$\bar{n}_B = \frac{l}{z}\delta_{Bz}$$

Compute $\frac{d}{d\lambda}(n_A U^A) := \bar{U}^B \bar{\nabla}_B (n_A U^A) = \bar{U}^B \bar{\nabla}_B (\bar{n}_A \delta U^A)$

$$0 = U^A \nabla_A U^B \rightarrow 0 = \bar{n}_B U^A \nabla_A U^B$$

$$0 = \bar{n}_B \left(\delta U^A \bar{\nabla}_A \bar{U}^B + \bar{U}^A \delta\Gamma^B_{AC} \bar{U}^C + \bar{U}^A \bar{\nabla}_A \delta U^B \right)$$

(2) (3) (4)

Compute (2) and (3) to solve for (4) = -(2) – (3).

$$0 = \bar{\nabla}_B (\bar{n}_A \bar{U}^A) = \bar{U}^A \bar{\nabla}_B (\bar{n}_A) + \bar{n}_A \bar{\nabla}_B (\bar{U}^A) \quad (2) = -(1)$$

$$\frac{d}{d\lambda}(n_A U^A) = (1) + (4) = 2\ (1) - (3) \ \& \ \delta\Gamma^z_{\alpha\beta} = -\frac{d-2}{2} \frac{l^{d-1} z^{d-1}}{16\pi d G_{d+1}} T_{\alpha\beta} + \ldots$$

$$\frac{d}{d\lambda}(n_A U^A) = d(1) = \cancel{d} \left(\frac{z}{l}\right)^{d-2} \frac{8\pi G_{d+1}}{\cancel{d}} T_{\alpha\beta} \bar{U}^\alpha \bar{U}^\beta + \ldots$$

Entropy

$$\frac{d}{d\lambda}S_{CFT} = \frac{1}{4G_{d+1}}\frac{d}{d\lambda}A_{ren} = \overbrace{\frac{1}{4G_{d+1}}\frac{d}{d\lambda}A_{loc\ exp}}^{\geq 0} + \frac{1}{4G_{d+1}}\frac{d}{d\lambda}A_{flow\ to\ \partial M}$$

Let $\sqrt{\sigma}$ be the volume element on any cut of $\partial\mathcal{H}$
Cut of bndy horizon, has dim = d-2.

$$\frac{d^2}{d\lambda^2}S_{CFT\ flow} = \frac{1}{4G_{d+1}}\frac{d^2}{d\lambda^2}A_{flow\ to\ \partial M}$$

$$= \frac{1}{4G_{d+1}}\left(\frac{z}{l}\right)^{(d-2)}\sqrt{\sigma}\frac{d}{d\lambda}(n_A U^A) = 2\pi T_{\alpha\beta}\bar{U}^\alpha \bar{U}^\beta + \ldots$$

Take z → 0. Higher corrections vanish.

Up to a (desired!) factor of -4G_d, this is precisely the source term in the Raychaudhuri equation that would arise if we couple the CFT to dynamical gravity! So for same BC:

$$\frac{d}{d\lambda}S_{total} = \frac{d}{d\lambda}S_{Bndy\ BH} + \frac{d}{d\lambda}S_{CFT} \geq \frac{d}{d\lambda}\frac{A_{Bndy\ BH}}{4G_d} + \frac{d}{d\lambda}S_{CFT\ flow} = 0$$

Graviton contributions handled as in Wall 2011.

$R^{(0)}{}_{ab} \neq 0$ (tentative)

You might expect that it's trivial to include the Ricci terms since they appear in sub-leading corrections:

$$g_{\alpha\beta}(z) = g^{(0)}{}_{\alpha\beta} + z^2 g^{(2)}{}_{\alpha\beta} + \ldots + z^d\ \bar{g}^{(d)}{}_{\alpha\beta} + z^{\mathrm{d}}\left(\frac{16\pi G_{d+1}}{d l^{d-1}}\right) T_{\alpha\beta} + \ldots$$

But there is an annoying factor of z^{-2} (from two Christoffels $\bar{\Gamma}^{z}{}_{\alpha\beta}$) in components of δU orthogonal to both $\bar{U}$ and $\bar{n}$ so the $g^{(2)}{}_{\alpha\beta}$ correction turns out to matter.

On the other hand, all interesting terms appear contracted with $\bar{U}$.
So we need only worry about

$$g^{(2)}{}_{\alpha\beta}\ \bar{U}^{\alpha}\ \delta U_{\perp}^{\beta} \propto R_{\alpha\beta}\ \bar{U}^{\alpha}\ \delta U_{\perp}^{\beta}$$

This vanishes if the boundary metric satisfiess NCC (NEC).

Turn off bndy gravity?

- Before setting $G_d = 0$ use physical process 1^{st} law to write $\delta S_{Bndy\ BH} = \frac{\delta E_{Bndy\ BH}}{T}$.
- Yields

$$0 \leq \delta S_{total} = \frac{\delta E_{Bndy\ BH}}{T} + \delta S_{CFT}$$

$$= -\frac{\delta E_{CFT}}{T} + \delta S_{CFT} = -\frac{1}{T}\delta F_{CFT}$$

So $\delta F_{CFT} \leq 0$.

I.e., the event horizon of a bulk black hole which ends on a bndy Killing horizon $\partial\mathcal{H}$ satisfies

$$\delta E_{bulk\ BH} - T\delta S_{ren}\,(Bulk\ BH) \leq 0.$$

2^{nd} law for black funnels/droplets.

Summary and Open Questions

- 1st order, perturabtive GSL for holographic CFTs.
  Also 2nd law for funnels/droplets.
- Coarse-grained GSL using Causal Holographic Info (CHI) as S_{coarse}. Supports coarse-grained S interp of CHI.
- What is CHI in the CFT?
  Kelly-Wall proposal: One-point entropy.
- Is there a corresponding fine-grained GSL based on HRT?
  This would be a *quantitative* test of HRT!
  How to derive it? What tools are available?
- Relation to recent work by Bousso, Cassini, Fisher, & Maldacena?

Bonus track:

What is the range of validity for the 1st law of entanglement in holographic large N theories? (w/ Kevin Kuns and Will Kelly)

$$S(\rho + \delta\rho) = \delta H_{modular} + \delta_2 S + \ldots$$

What is the relative size of consecutive terms?
For geometric $H_{modular}$, bulk says higher terms are suppressed by $G_N \sim N^{-2}$ for $\delta\rho$ geometric in bulk. How to see this in CFT?

We can show this (w/ interesting coefficient) via a CFT argument in a toy model:

Take $\rho_{total} = \otimes_{i=1}^{N^2} \rho_{dof}$

$$\delta\rho_{total} = \sum\nolimits_{\sigma\ positions} \rho_{dof} \otimes \rho_{dof} \otimes \ldots \otimes \sigma \otimes \ldots \otimes \rho_{dof} \otimes \rho_{dof}$$

Produced at leading order by Unitaries built from single-trace operators & tracing over complementary region.
Result generalizes Marolf, Minic, & Ross.

Can we generalize this to better models of SU(N) SYM states?

More Open Questions

- How much do we really understand about RT/HRT?
- To what extent is LM a "derivation?"
- Are complex extremal surfaces relevant?
  (See related talk by Sebastian Fischetti)
- If not, why not? (Very relevant for geodesic approx to 2 pt functions, even in AdS_{2+1}.)
- If so, what is the corresponding entropy? $S = \mathrm{Re}\, A$?