Holographic Entanglement Entropy from 2d CFT

Tom Hartman
Cornell University

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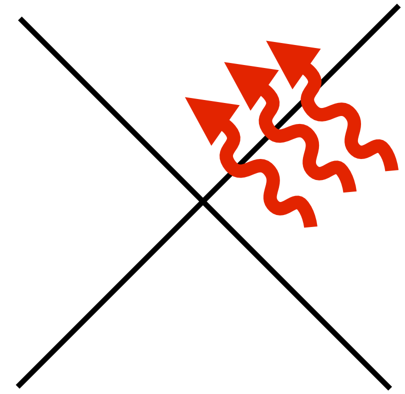
Horizons in general relativity obey universal thermodynamic laws.

- First law
- 2nd law
- etc...



These can be viewed as IR constraints on the UV completion of quantum gravity, required by diffeomorphism invariance:

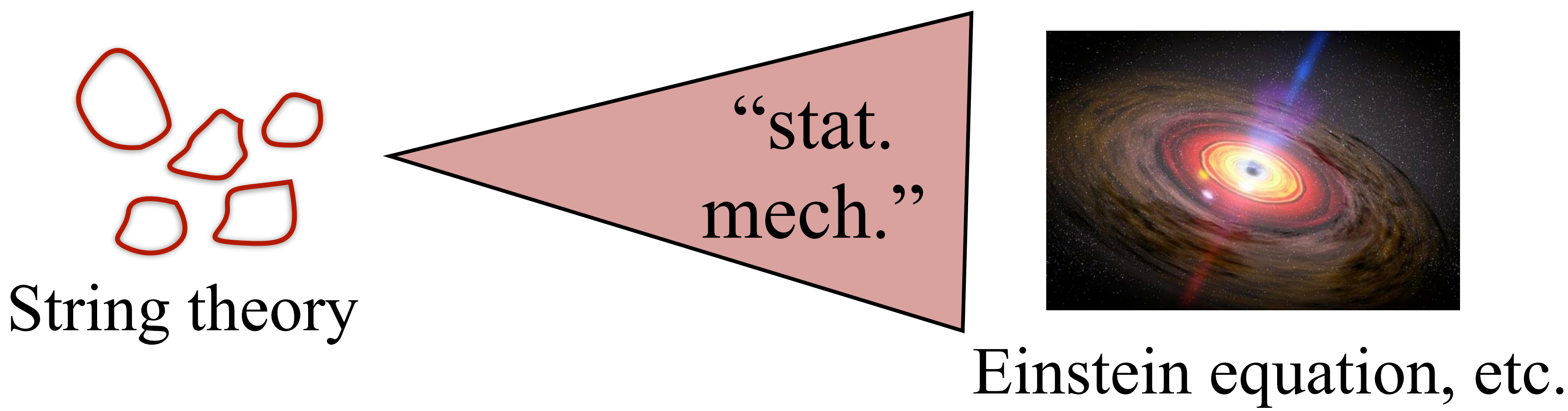
- log (# states) ~ area of black hole horizon
- $\eta/s \sim \dfrac{1}{4\pi}$
- Entanglement entropy = Ryu-Takayanagi formula
- etc...

We do not have any decent microscopic understanding of where these universal laws come from.

- Holography suggests that we should try to understand these microscopic laws from some large class of CFTs.
- This would be a step towards deriving (some corner of) AdS/CFT.

To derive universal features, the Lagrangian of the dual CFT is probably the wrong starting point.

We need an effective description that does not rely on every microscopic detail.



'Sparse CFT' approach

- There are (at least) two prerequisites for a CFT to act like quantum gravity in the semiclassical regime:
  - large N (large central charge c)
  - Small number of low-dimension operators
- Within this class, use general properties of CFT --- crossing, modular invariance, OPE, etc --- to derive universal behavior

But...

- How sparse is sparse enough?
- Are further assumptions needed to get a CFT with a holographic dual?
- So far, most concrete results are in 2d CFT where we the power of Virasoro symmetry. Can this approach be applied in d > 2?

This talk: Sparse CFTs in 2d

I. Spectrum and thermodynamics = BTZ black holes

TH, Christoph Keller, Bogdan Stoica '14

II. Holographic entanglement entropy

in vacuum: TH '13

in excited states: Curtis Asplund, Alice Bernamonti, Federico Galli, TH (to appear)

Objection: But 3d gravity is trivial!

- It has no propagating d.o.f.
- However, it is nontrivial if we include topology, defects, or matter.

Similarly, in 2d CFT many things are completely fixed by Virasoro symmetry:

- <TT...> correlators
- Cardy entropy at high energy
- Entanglement entropy of a single interval $S_A = \frac{c}{3} \log \ell$

We will always be including some nontrivial topology or defects. None of the results I'll discuss are consequences of Virasoro alone.

From the entropy of the BTZ black hole, the spectrum of 3d quantum gravity is

$$S(\Delta) = O(1) \qquad (\Delta < \frac{c}{12})$$

$$S(\Delta) = 2\pi\sqrt{\frac{c}{6}(\Delta - \frac{c}{12})} \qquad (\Delta > \frac{c}{12})$$

In the canonical ensemble,

$$\log Z \sim \frac{c\beta}{12} \qquad (\beta > 2\pi)$$

$$\log Z \sim \frac{\pi^2 c}{3\beta} \qquad (\beta < 2\pi)$$

with a sharp Hawking-Page transition at $\beta = 2\pi$

- This was derived from the dual D1-D5 CFT by Strominger and Vafa ('95). But the calculation relied on the microscopic details -- why is the answer universal?

- In '97, Strominger computed the same entropy using the Cardy formula. This argument is much more universal.

$$S_{Cardy}(\Delta) = S_{BTZ}(\Delta) = 2\pi\sqrt{\frac{c}{6}(\Delta - \frac{c}{12})} \qquad (\Delta \gg c)$$

- However, in a general CFT the Cardy formula applies only as

$$\Delta \to \infty$$

- In holographic theories (including D1-D5), the Cardy formula should apply whenever the black hole exists:

$$\Delta > \frac{c}{12}$$

In 2d, the extended regime of validity of the Cardy formula is a key feature that distinguishes holographic CFTs from the rest.

What theories have this property?

We showed that at large c, modular invariance guarantees that CFT reproduces BTZ thermodynamics if and only if the low-lying density of states is bounded by

$$\rho(\Delta) \leq e^{2\pi\Delta} \qquad (\Delta < \frac{c}{12})$$

This might be a reasonable definition of a “sparse” CFT in 2d; it is enough to have black-hole-like behavior, but we do not know if it is enough to behave like quantum gravity in all respects.

TH, Christoph Keller, Bogdan Stoica ’14

Philosophy: Large c + gap ==> only the vacuum contributes to

$$\sum e^{-\beta E}$$

A similar philosophy will be used to derive the holographic entanglement formula.

Entanglement entropy

$$S_A = -\mathrm{tr}\ \rho_A \log \rho_A$$

In 1+1 dimensions:

Space is a line, so **A** consists of one or more intervals:

Replica Method

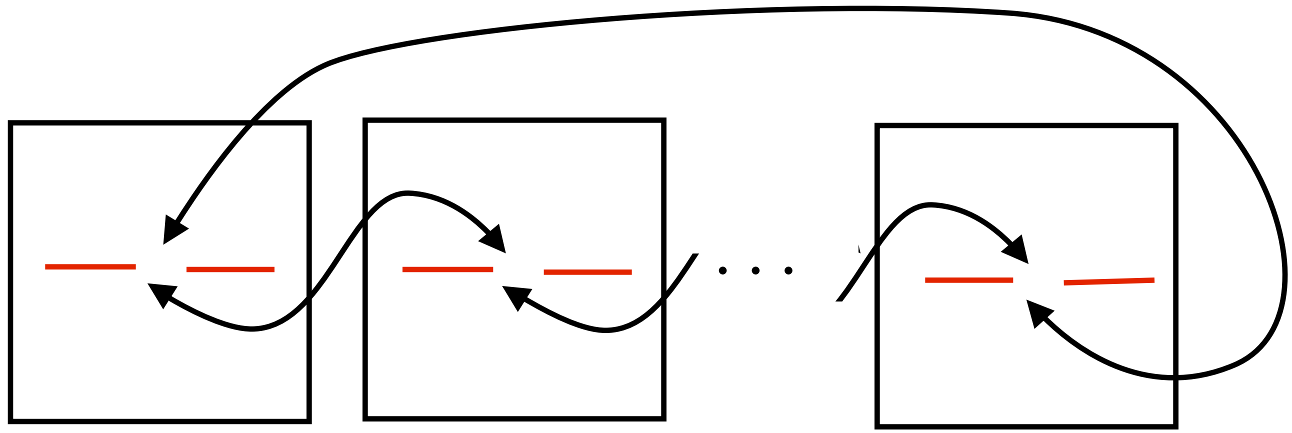
$$Z_n = \mathrm{Tr}\ \rho_A^n$$

$$S_A = \frac{1}{1-n} \log Z_n \Big|_{n \to 1}$$

The replica method is useful in states that can be prepared by a path integral.

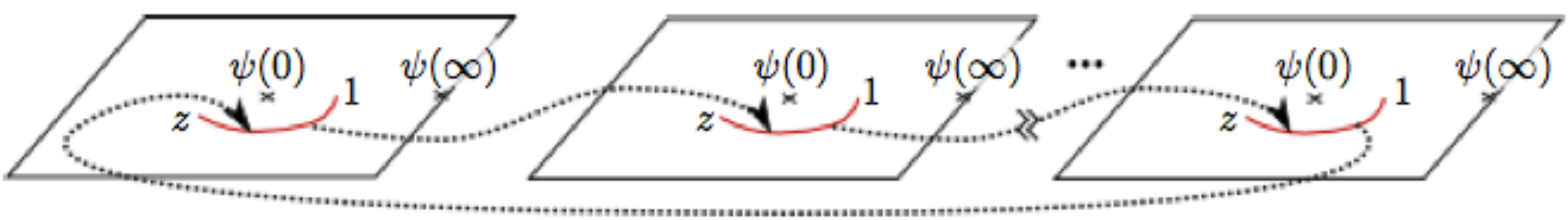
Example #1: Two intervals in vacuum on line

$$\mathrm{Tr}\ \rho_A^n =$$



Example #2: One interval in a primary state $|\psi\rangle$ in radial quant.

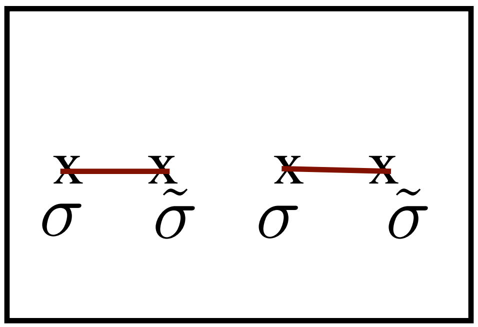
$$\mathrm{Tr}\ \rho_A^n =$$



Our goal is to compute these partition functions in sparse CFTs.

First, recast as correlation functions involving ‘twist operators’:

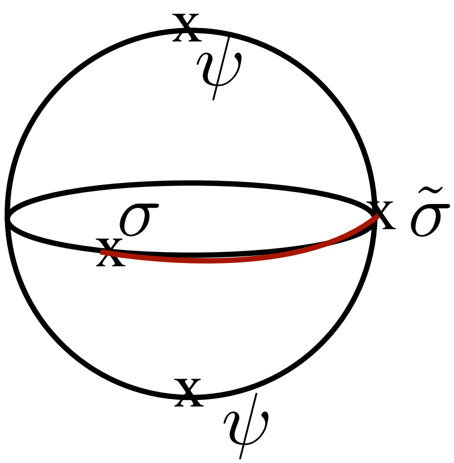
Two intervals in vacuum:



$$\langle 0|\sigma\tilde{\sigma}\sigma\tilde{\sigma}|0\rangle$$

Headrick ’10
TH ’13

One interval in primary state:



$$\langle\psi|\sigma\tilde{\sigma}|\psi\rangle = \langle 0|\psi\sigma\tilde{\sigma}\psi|0\rangle$$

Asplund, Bernamonti, Galli, TH (to appear)

These are ordinary (local) correlation functions in a cyclic orbifold theory with *n* copies of the original CFT, CFT^n/Z_n

So we can use the conformal block expansion:

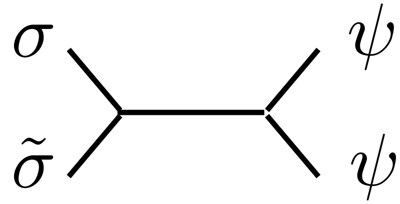
$$\langle \psi \sigma \tilde{\sigma} \psi \rangle = \sum_{\Delta} c_{\Delta}^2 \mathcal{F}(\Delta, z) \mathcal{F}(\bar{\Delta}, \bar{z})$$

Virasoro Conformal Blocks

OPE coefficient

$$H_n = \frac{c}{24}(n - 1/n) = \text{ dimension of twist operator}$$

Expand in the channel:



Universality?

- In general, these entanglement entropies are not universal: they depend on the operator content and OPE coefficients of the CFT

- However, in all holographic theories they should be computed by the length of a geodesic

- So we expect a universal answer in sparse CFT

Outline of the large-c calculation

Virasoro blocks have a nice form at large central charge:

$$\mathcal{F}(\Delta, z) \approx e^{-cf(\frac{\Delta}{c}, z)}$$

Zamolodchikov '87

Thus we expect that the OPE sum is dominated by a saddle. In sufficiently sparse CFTs, the "saddle" must land on the vacuum rep:

$$\mathrm{Tr}\ \rho_A^n \approx e^{-cf(0,z)-cf(0,\bar{z})}$$

Comments:

- This contribution is universal (independent of CFT details)
- Leading order in $1/c$ (but all orders in OPE!)
- It is the Virasoro block for the vacuum rep, which in 2d includes everything you can make out of stress tensors:

$$1\ , T\ , \partial T\ , T^2\ , T\partial T\ , \cdots$$

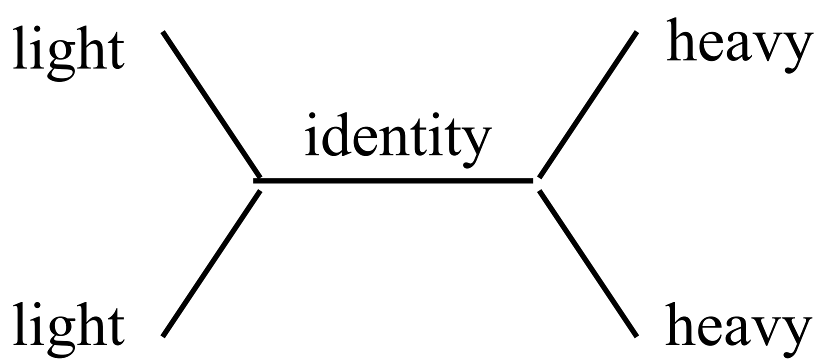
- Heavy correlators are exponentially dominated by exchange of operators built from the stress tensor. (Dual: 3d graviton)

Entanglement entropy

- This is our formula for the Renyi entropy. In general, the block f can only be computed as a series expansion.
- But for *n-->1*, twist operators become light

$$H_n \sim \frac{c}{12}(n-1) + \cdots$$

- So for EE, we just need the Virasoro block for the case:



- This is known! Fitzpatrick, Kaplan, Walters '14

$$f = \frac{12H_n}{c}\log\left(\frac{1-z^{\alpha_\psi}}{\alpha_\psi}\right) + \frac{6H_n}{c}(1-\alpha_\psi)\log z \ , \qquad \alpha_\psi \equiv \sqrt{1-\frac{24h_\psi}{c}}$$

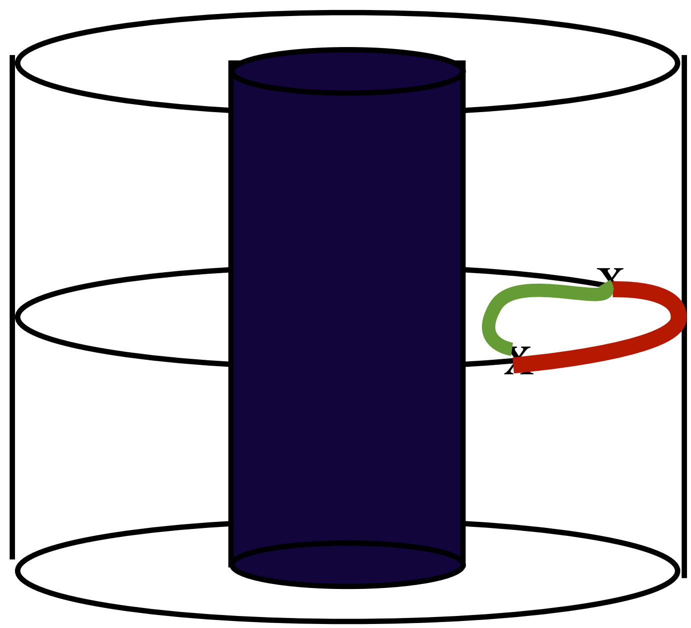
This gives a closed-form answer for the entanglement entropy in a primary state:

$$S = \frac{c}{6} \log \frac{(z\bar{z})^{\frac{1}{2}(1-\alpha_\psi)}(1-z^{\alpha_\psi})(1-\bar{z}^{\alpha_\psi})}{\alpha_\psi^2} + \text{conformal factors}$$

- This agrees with holographic formulas. I'll give some examples/applications.
- There are some caveats about when this CFT calculation is reliable/justified, which I'll mention in a few slides.
- Since primaries + conformal transformations give a complete basis of states, it seems plausible that this is close to a complete microscopic derivation of the RT/HRT/LM/etc formula in AdS_3 from CFT, in an arbitrary state.

Application #1: Black Holes and Conical Defects

To apply our EE formula to a heavy state on a cylinder, set



$$z = e^{i\ell}, \quad \bar{z} = e^{-i\ell}$$

$$S_A = \frac{c}{3} \log \left[\frac{\beta_\psi}{\pi} \sinh\left(\frac{\pi \ell}{\beta_\psi} \right) \right]$$

$$\beta_\psi \equiv \frac{2\pi}{\sqrt{24 h_\psi / c - 1}}$$

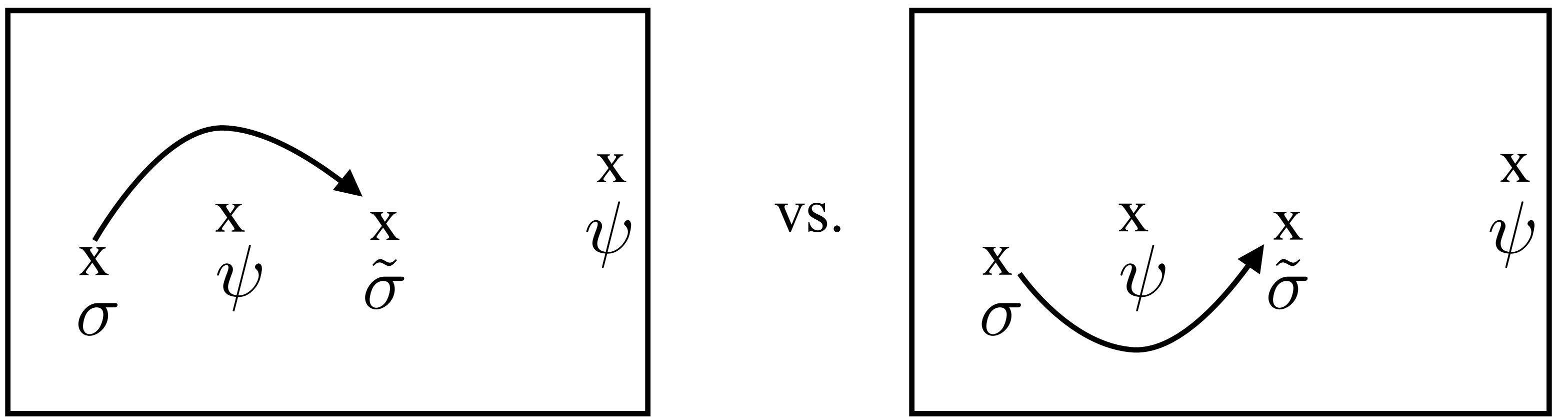
- Agrees with geodesic on the BTZ black hole of temperature β_ψ
- *[Aside: not fixed by conformal symmetry, since we are on a circle]*

The non-pertubative subtlety:

- The CFT calculation was reliable in some finite neighborhood of

$$\sigma \longrightarrow \tilde{\sigma}$$

- It must fail for some z, when the identity block in another channel dominates:



- In the original OPE, this other channel must appear as terms non-perturbative in *1/c*. Saddles exchange dominance in z plane.
- The caveat: We cannot prove that we've accounted for all non-perturbative contributions. We simply assume that the full, non-perturbative answer is given by the dominant identity block.
  - Does this require further restrictions on definition of "sparse"?

Back to the black hole:

- Previously I wrote the formula in a single OPE channel.
- The full answer (with assumption just stated) is therefore

$$S_A = \frac{c}{3} \log \left[\frac{\beta_\psi}{\pi} \sinh \left(\frac{\pi \ell_{min}}{\beta_\psi} \right) \right]$$

$$\ell_{min} = \min(\ell, 2\pi - \ell)$$

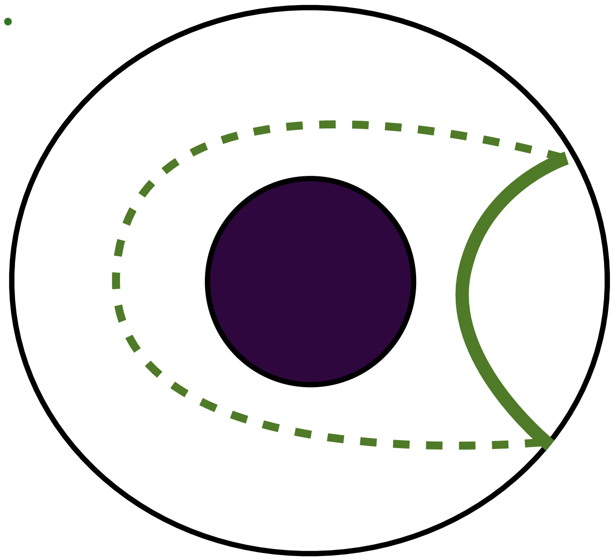
The CFT calculation was in a pure state,

$$\rho = |\psi\rangle\langle\psi|$$

So this is actually *not* dual to BTZ but to a BTZ microstate.

And indeed, $S_A = S_{A^c}$

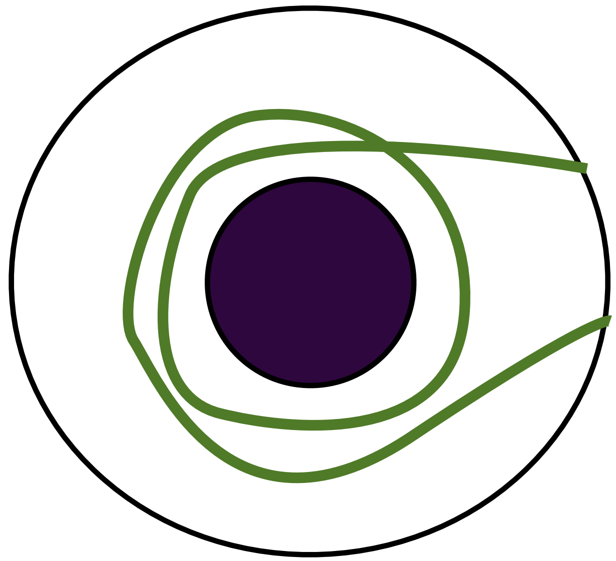
The two different OPE channels correspond to two different geodesics on BTZ:



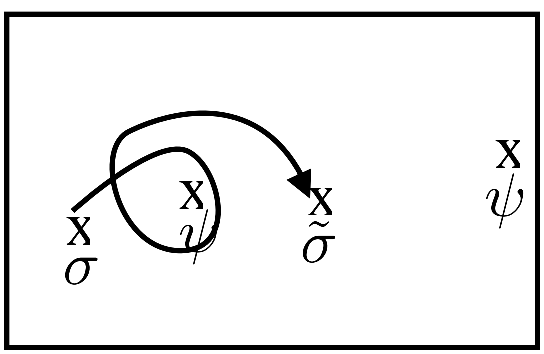
Recall in eternal BTZ the holonomy condition requires a disconnected geodesic, wrapping the horizon.

- Makes no appearance in the CFT calculation.
- So the holonomy condition is *not* imposed in this case.
- This makes sense, since horizon should not contribute in a pure state.

Multiply-wrapped geodesics?



- In the CFT, these correspond to the identity rep in a bizarre choice of OPE channel:



$$\langle\psi\sigma\tilde{\sigma}\psi\rangle = \text{identity} + \cancel{\text{much bigger stuff}}$$

- This suggests that if we are to make sense of non-minimal surfaces, we need some notion of "microcanonical" rather than "canonical" Renyi EE.

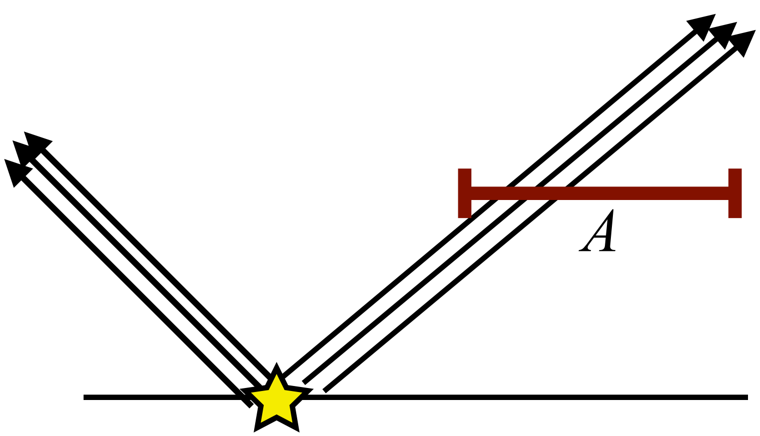
Application #2: Local quantum quench

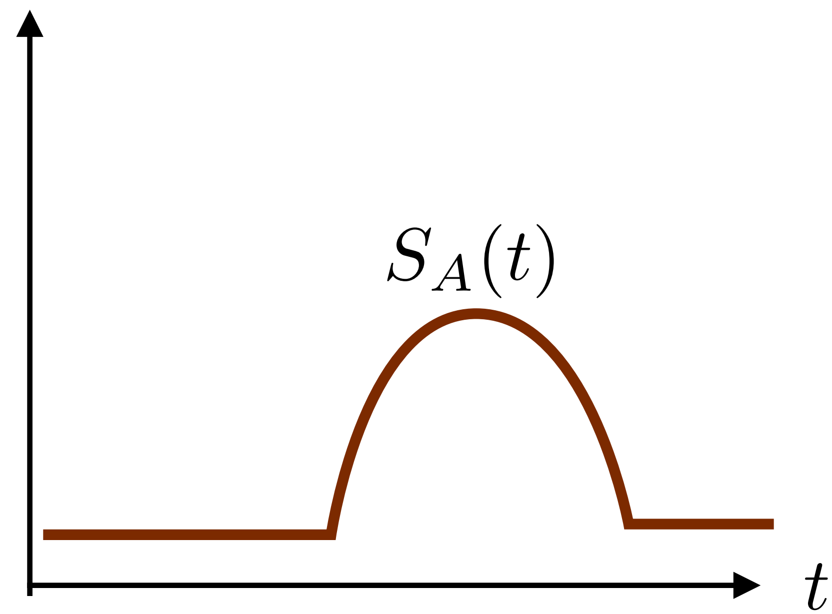
- "Quench": sudden change external to the system, then evolve under the usual Hamiltonian
- We consider a "local operator quench" where we insert a local operator at $x = 0, \quad t = i\delta$

$$|Q\rangle = \psi(i\delta)|0\rangle$$

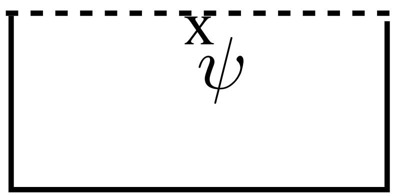
Nozaki, Numasawa, Takayanagi '14; Caputa, Nozaki, Takayanagi '14

Lorentzian cartoon

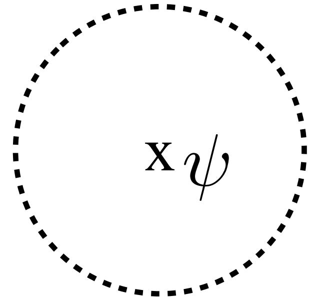




This is different from a primary state,



vs.



But is related by a conformal transformation.

The entanglement Renyi correlator is

$$
\begin{aligned}
Z_n &= \langle \psi(-i\delta, 0)\sigma(\ell_1, t)\tilde{\sigma}(\ell_2, t)\psi(i\delta, 0)\rangle \\
&= (\text{conformal prefactor}) \times \langle \psi(0)\sigma(z, \bar{z})\tilde{\sigma}(1)\psi(\infty)\rangle
\end{aligned}
$$

This is the correlator we computed before,
but now in Lorentzian signature.

Therefore

$$S = \frac{c}{6} \log \frac{(z\bar{z})^{\frac{1}{2}(1-\alpha_\psi)}(1 - z^{\alpha_\psi})(1 - \bar{z}^{\alpha_\psi})}{\alpha_\psi^2} + \text{conformal factors}$$

where z is the cross ratio.

But if we evaluate z we find

$$z = 1 + O(\delta)$$

Naively the answer is constant in time!

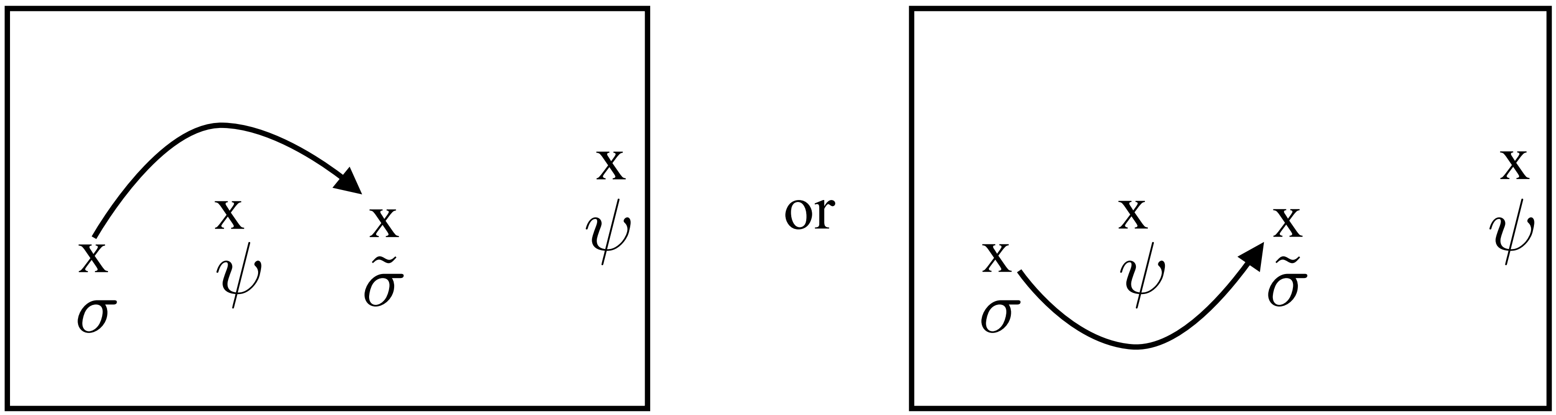
But not all 1's are created equal. There are branch cuts in this expression and we can take

$$z \to e^{\pm 2\pi i} z, \qquad \bar{z} \to e^{\pm 2\pi i} \bar{z}$$

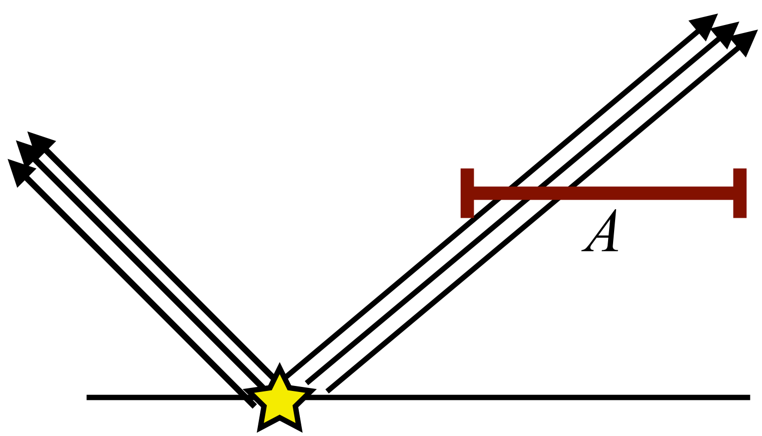
independently.

This is a 2-step process:

First, choose a Euclidean OPE channel:



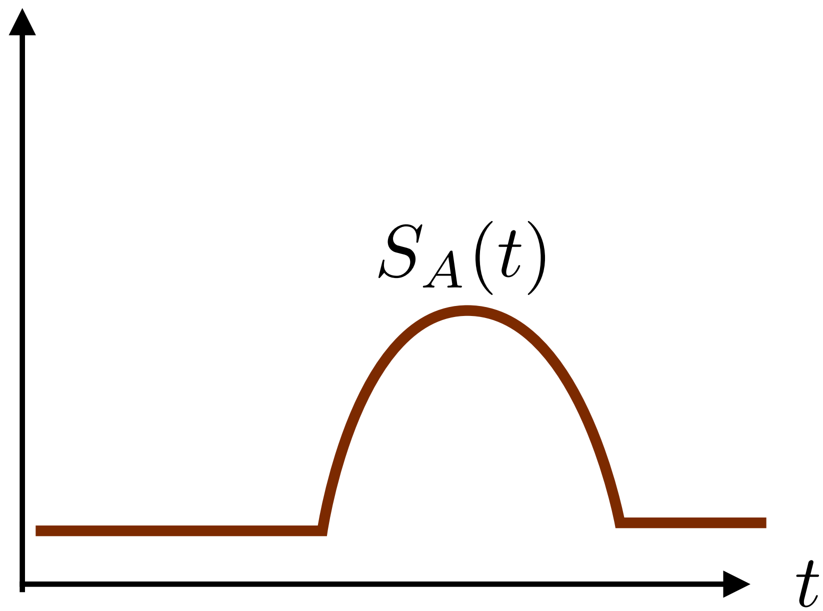
Second, in Lorentzian signature, z or $\bar{z}$ crosses a branch cut when one operator passes through the light-cone of another.



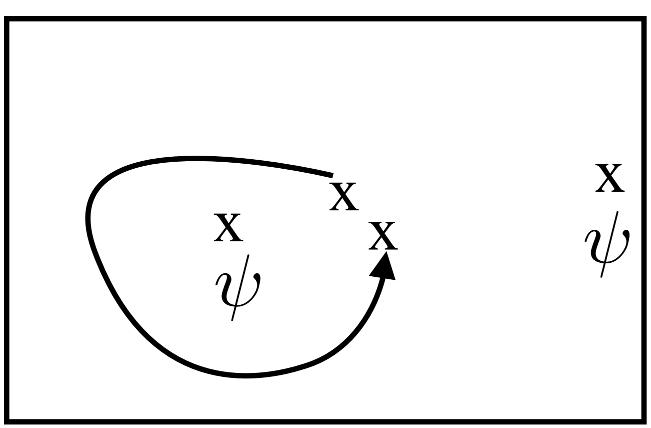
How to cross these branch cuts is a dictated by the operator ordering:

$$\langle \psi \sigma \tilde{\sigma} \psi \rangle$$

After keeping track of these branch cuts, we find that the shape and height of the entanglement bump:



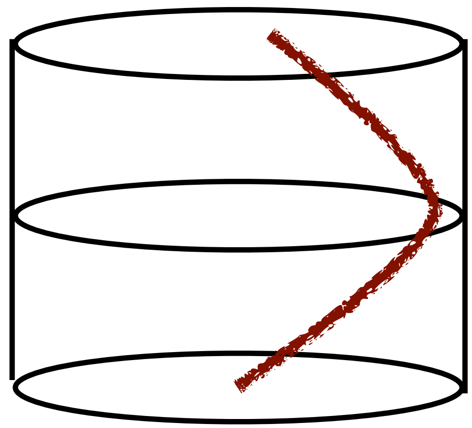
are related to the braiding of the Virasoro vacuum block:



$$\mathcal{F}(e^{2\pi i}z) \text{ as } z \to 0$$

Comparison to holography

- A holographic model of a local quench was proposed by Nozaki, Numasawa, and Takayangi '13: An infalling particle geometry



- This is the holographic dual of our calculation. The infalling particle hits the boundary at imaginary time $t = i\delta$

- CFT results agree precisely with geodesic lengths on this background.

Conclusion

- Sparse CFTs in two dimensions have universal, gravity-like behavior.
- A sparse CFT has large c and a restricted number of light states,

$$\rho(\Delta) \leq e^{2\pi\Delta} \qquad (\Delta < \frac{c}{12})$$

- We also "assumed away" certain contributions to the OPE; is this automatic or does this impose additional requirements on sparse CFT?

Homework

d-dimensional CFTs with properties X,Y,Z have the thermodynamics, entanglement entropy, and viscosity-to-entropy ratio of d+1-dimensional Einstein gravity.

- Find X,Y,Z
- Derive this