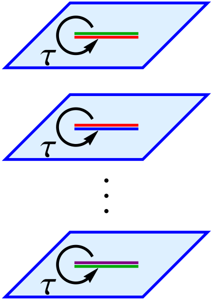
Holographic Entanglement and Renyi Entropies



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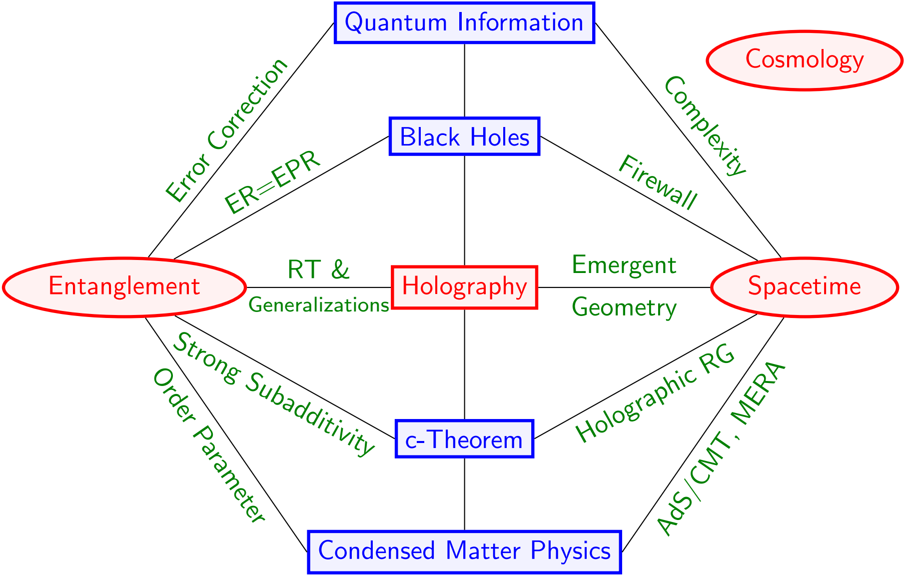
August 21, 2014

Based on
[XD 1409.????]
[XD 1310.5713]
[Barrella, XD, Hartnoll & Martin 1306.4682]

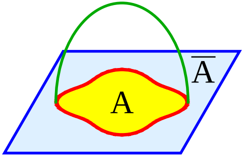
Quantum Information in Quantum Gravity, University of British Columbia

Entanglement and Spacetime



Holographic Entanglement Entropy

A remarkably simple prescription in QFTs dual to Einstein gravity:



$$S_A = \frac{(\text{Area})_{\text{min}}}{4G_N}$$

[Ryu & Takayanagi '06]

Holographic Entanglement Entropy

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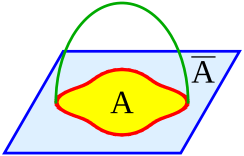


$$S_A = \frac{(\text{Area})_{\min}}{4G_N}$$

[Ryu & Takayanagi '06]

- Satisfies strong subadditivity. [Headrick & Takayanagi '07]
- Reproduces exact results for one interval in 1+1D CFTs. [Holzhey, Larsen & Wilczek '94; Calabrese & Cardy '04]
- First derived for spherical entangling surfaces. [Casini, Huerta & Myers '11]
- Proven for 2D CFTs with large c. [Hartman 1303.6955; Faulkner 1303.7221]
- Derived generally for Einstein gravity. [Lewkowycz & Maldacena 1304.4926]
- Bulk one-loop corrections: [Barrella, XD, Hartnoll & Martin 1306.4682] [Faulkner, Lewkowycz, & Maldacena 1307.2892; Engelhardt & Wall 1408.3203]
- Higher spin gravity: [Ammon, Castro & Iqbal 1306.4338; de Boer & Jottar 1306.4347]
- Bulk EOMs from EE first law: [Lashkari et al. 1308.3716; Faulkner et al. 1312.7856]

Outline

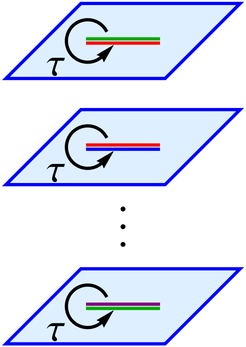
1. Holographic Replica Trick
2. Entanglement Entropy for Higher Derivative Gravity
3. Universal Terms in Holographic Renyi Entropy
4. Conclusion and Open Questions

Replica Trick

Introduce Rényi entropy:

$$\boxed{S_n = -\frac{1}{n-1}\log \mathrm{Tr}\,\rho^n} \qquad \Rightarrow \qquad S_{EE} = \lim_{n\to 1} S_n = -\mathrm{Tr}\,\rho \log \rho$$

At integer n, Rényi entropy can be written in terms of partition functions:



E.g. 1+1D QFT

$$\boxed{S_n = -\frac{1}{n-1}\left(\log Z[M_n] - n \log Z[M_1]\right)}$$

- M_1: original (Euclidean) spacetime manifold.
- M_n: n-fold cover $=$ n copies of M_1 glued together along A in cyclic order.
- τ: angle around ∂A, range extended to $2\pi n$.
- n-fold cover does not make much sense for non-integer n.
- Holographic dual side provides much "better" analytic continuation. [Lewkowycz & Maldacena]

Holographic Dual of the n-Fold Cover

Build a bulk solution B_n whose boundary is M_n:

$$Z[M_n] = e^{-S[B_n]} + \cdots$$

Basic idea

1. Use gauge-gravity duality to calculate $S[B_n]$.
2. Analytically continue it to non-integer n.
3. Expand to $O(n-1)$ to extract EE.

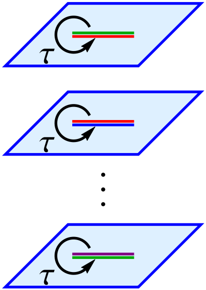
Very complicated in general, can be explicitly worked out only in special cases e.g. AdS_3/CFT_2. [Faulkner 1303.7221; Barrella, XD, Hartnoll & Martin 1306.4682]

But...

- We do not need B_n explicitly.
- For EE, only need $S[B_n]$ near $n \approx 1$: $\boxed{S_n = -\frac{1}{n-1}\left(S[B_n] - nS[B_1]\right)}$
- If we can find a family of bulk configurations interpolating between integer n, then we can expand in $n-1$!

Replica Symmetry

The n-fold cover has Z_n symmetry: $\tau \to \tau + 2\pi$.



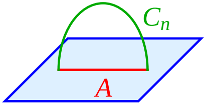
Assume: Z_n symmetry extends to the bulk B_n.

Agrees with e.g. [Faulkner 1303.7221]

Then consider the orbifold: $\hat{B}_n = B_n/Z_n$

- Regular except at fixed points.
- Fixed points form codimension 2 surface C_n.
- C_n: conical defect with opening angle $2\pi/n$, anchored at ∂A: $ds^2 = \rho^{-2(1-\frac{1}{n})}(d\rho^2 + \rho^2 d\tau^2) + \cdots$

Z_n quotient



How does this help us calculate EE?

By construction: $S[B_n] = nS[\hat{B}_n]$ at integer n

$$\Rightarrow \quad S_n = \frac{n}{n-1}\left(S[\hat{B}_n] - S[\hat{B}_1]\right)$$

Note: $S[\hat{B}_n]$ does not include contributions from C_n.

Now plausible that we can analytically continue $\hat{B}_n$.

Analytic Continuation of the Orbifold $\hat{B}_n$

There are two equivalent methods.

1. "Boundary condition" method

Solve all EOMs and demand the metric near C_n as

$$ds^2 = \rho^{-2\epsilon}(d\rho^2 + \rho^2 d\tau^2) + (g_{ij} + 2K_{aij}x^a)dy^i dy^j + \cdots$$

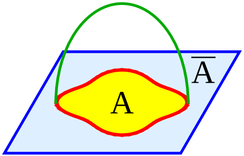
- An unconventional "IR" boundary condition.
- Justified by considering integer n and impose Z_n symmetry.
- In general has conical defect with deficit $2\pi\epsilon = 2\pi\left(1 - \frac{1}{n}\right)$.

2. "Cosmic brane" method

Replace C_n by a codimension 2 brane! Solve all EOMs resulting from

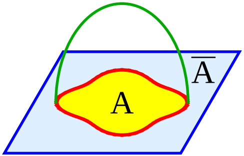
$$S_{\text{total}} = S_{EH} + S_B = -\frac{1}{8\pi G_N}\int d^D x\sqrt{G}R + \frac{\epsilon}{4G_N}\int d^d y\sqrt{g}$$

Cosmic branes are "straight" allowing us to glue $\hat{B}_n$ back to B_n for $n \in \mathbb{Z}$.



$$S_A = \frac{(\text{Area})_{\text{min}}}{4G_N} \Rightarrow S_A = \frac{(???)_{\text{min}}}{4G_N}$$

After all, string theory produces α' corrections.



$$S_A = \frac{(\text{Area})_{\text{min}}}{4G_N} \Rightarrow S_A = \frac{(???)_{\text{min}}}{4G_N}$$

After all, string theory produces α' corrections.

- Analogous to: Bekenstein-Hawking Entropy $\Rightarrow$ Wald Entropy for BHs:

$$S_{\text{Wald}} = -2\pi \int d^d y \sqrt{g} \frac{\partial L}{\partial R_{\mu\rho\nu\sigma}} \varepsilon_{\mu\rho}\varepsilon_{\nu\sigma}$$

[Wald '93]

- In general, S_{Wald} cannot be S_{EE}. [Hung, Myers & Smolkin '11]
- Even before Wald, there existed a different formula S_{JM} for BH entropy in Lovelock gravity. [Jacobson & Myers '93]
- They differ only by extrinsic curvature terms (=0 for Killing horizons).
- For Gauss-Bonnet, S_{JM} passes consistency checks as S_{EE}. [Hung, Myers & Smolkin '11]

Entropy Formula for Higher Derivative Gravity

General entropy formula for $L(R_{\mu\rho\nu\sigma})$: [XD 1310.5713]

$$S_{EE} = 2\pi \int d^d y \sqrt{g} \left\{ \underbrace{\frac{\partial L}{\partial R_{z\bar{z}z\bar{z}}}}_{\text{Wald's formula}} + \underbrace{\sum_\alpha \left(\frac{\partial^2 L}{\partial R_{zizj} \partial R_{\bar{z}k\bar{z}l}} \right)_\alpha \frac{8 K_{zij} K_{\bar{z}kl}}{q_\alpha + 1}}_{\text{"Anomaly" from extrinsic curvature}} \right\}$$

- Encompasses previous results of special cases (e.g. giving S_{JM} for Gauss-Bonnet): [Fursaev, Patrushev, & Solodukhin 1306.4000; Chen & Zhang 1305.6767; Bhattacharyya, Sharma, & Sinha 1305.6694, 1308.5748; . . .]
- Can show minimization prescription for at least 3 classes of examples: f(R), Lovelock, general 4-derivative gravity.
- Covariant version exists.
- Although derived for entanglement entropy, this formula also applies for BH entropy, generalizing Wald's formula to non-stationary BHs.

$$S_{EE} = 2\pi \int d^d y \sqrt{g} \left\{ \underbrace{\frac{\partial L}{\partial R_{z\bar{z}z\bar{z}}}}_{\text{Wald's formula}} + \underbrace{\sum_\alpha \left(\frac{\partial^2 L}{\partial R_{zizj} \partial R_{\bar{z}k\bar{z}l}} \right)_\alpha \frac{8 K_{zij} K_{\bar{z}kl}}{q_\alpha + 1}}_{\text{"Anomaly" from extrinsic curvature}} \right\}$$

Outline for derivation

1. Calculate S_{on}(bulk with conical deficit)
2. Take small $n-1$ limit, conical deficit $\epsilon \approx n - 1$.
3. First-order variation of S_{on} localizes at defect: from either δ-function or potential logarithmic divergences:

$$R_{zizj} \sim \frac{\epsilon}{\rho} K_{zij} + \cdots \quad \Rightarrow \quad \delta S_{\text{on}} \propto \int \rho d\rho \frac{\epsilon^2}{\rho^2} \rho^{\# \epsilon} \sim \frac{\epsilon}{\#}$$

Logarithmic Terms Are Universal

In even-dimensional CFTs, certain logarithmically divergent terms are universal, i.e. they do not depend on much of the theory besides a few numbers such as anomaly coefficients.

- Partition function:

$$\log Z = (\text{power divergences}) + \log \epsilon \int d^d x \sqrt{g} \mathcal{A} + (\text{finite})$$

$$\mathcal{A}(d=2) = \frac{c}{24\pi} R\,, \qquad \mathcal{A}(d=4) = \frac{a}{16\pi^2} E_{(4)} - \frac{c}{16\pi^2} I_{(4)}$$

- Entanglement entropy across a codimension-2 surface Σ:

$$S_{EE}(d=2) \sim -\frac{c}{6} \,\text{Volume}\,(\Sigma) \log \epsilon$$

$$S_{EE}(d=4) \sim \log \epsilon \left[\frac{a}{2\pi} \int_\Sigma R_\Sigma + \frac{c}{2\pi} \int_\Sigma \left(\mathrm{Tr} K^2 - \frac{1}{2} (\mathrm{Tr} K)^2 - C^{ab}{}_{ab} \right) \right]$$

[Solodukhin '08]

Can derives these by PBH (Penrose–Brown–Henneaux) transformations.

Universal Terms in Renyi Entropies

Renyi entropies S_n

- Contain richer information about ρ than S_{EE}.
- Are interesting at special n: $n = 1/2$ (negativity), $n = 0$, $n \to \infty$.
- Have nice holographic interpretation in terms of cosmic branes.

They also have universal logarithmic terms in even dimensions.

$d = 2$

$$S_n \sim -\frac{c}{12}\left(1 + \frac{1}{n}\right) \text{Volume}\,(\Sigma) \log \epsilon$$

$d = 4$

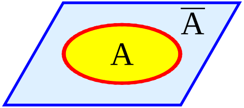
$$S_n \sim \log \epsilon \left[\frac{f_a(n)}{2\pi} \int_\Sigma R_\Sigma + \frac{f_b(n)}{2\pi} \int_\Sigma \left(\mathrm{Tr} K^2 - \frac{1}{2}(\mathrm{Tr} K)^2\right) - \frac{f_c(n)}{2\pi} \int_\Sigma C^{ab}{}_{ab}\right]$$

[Fursaev '12]

$$S_n \sim \log\epsilon \left[\frac{f_a(n)}{2\pi} \int_\Sigma R_\Sigma + \frac{f_b(n)}{2\pi} \int_\Sigma \left(\mathrm{Tr} K^2 - \frac{1}{2} (\mathrm{Tr} K)^2 \right) - \frac{f_c(n)}{2\pi} \int_\Sigma C^{ab}{}_{ab} \right]$$

$f_a(n)$ is computed by considering a spherical Σ in flat space:

- The n-fold cover may be conformally mapped to a hyperboloid $H^3 \times S^1$, with the size of S^1 being $\beta = 2\pi n$. [Casini, Huerta & Myers '11]



$\Rightarrow$

- $f_a(n)$ is completely determined by $\log Z[H^3 \times S^1] \propto \text{Volume}\,(H^3)$.
- This can be computed holographically as the dual geometry is a hyperbolic black hole.

$$S_n \sim \log \epsilon \left[\frac{f_a(n)}{2\pi} \int_\Sigma R_\Sigma + \frac{f_b(n)}{2\pi} \int_\Sigma \left(\mathrm{Tr} K^2 - \frac{1}{2} (\mathrm{Tr} K)^2 \right) - \frac{f_c(n)}{2\pi} \int_\Sigma C^{ab}{}_{ab} \right]$$

What about $f_b(n)$ and $f_c(n)$?

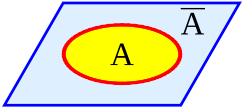
- Not much was known about them until [Lewkowycz & Perlmutter 1407.8171] proposed that $f_c(n)$ may be derived from $f_a(n)$:

$$f_c(n) = \frac{n}{n-1} \left[a - f_a(n) - (n-1) f_a'(n) \right] .$$

- It has also been conjectured that $f_b(n) = f_c(n)$. [Lee, McGough & Safdi 1403.1580]
- I will propose a holographic derivation of these relations.

Deformed Hyperboloid

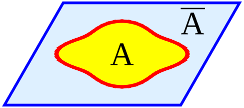
Similar to how we map the spherical case to a hyperboloid $H^3 \times S^1$:



$\Rightarrow$

$$ds_4^2 = \frac{1}{\rho^2}\left[\frac{d\rho^2}{1+\rho^2} + g_{ij}dy^i dy^j\right] + d\tau^2\,, \qquad g_{ij}dy^i dy^j = d\Omega_2^2$$

We can map the case of arbitrary Σ in arbitrary background to a deformed hyperboloid:



$\Rightarrow$

$$\boxed{\begin{aligned} ds_4^2 = \frac{1}{\rho^2}&\left[\frac{d\rho^2}{1+\rho^2} + \left(g_{ij} + 2K_{aij}x^a + Q_{abij}x^a x^b\right) dy^i dy^j\right] \\ &+ (1+T\rho^2)d\tau^2 + 2U_i d\tau dy^i + (\text{higher orders})\,, \qquad x^{1,2} \equiv \rho e^{\pm i\tau} \end{aligned}}$$

Partition Function on Deformed Hyperboloid

$$ds_4^2 = \frac{1}{\rho^2}\left[\frac{d\rho^2}{1+\rho^2} + \left(g_{ij} + 2K_{aij}x^a + Q_{abij}x^a x^b\right) dy^i dy^j\right] + (1+T\rho^2)d\tau^2 + 2U_i d\tau dy^i + (\text{higher orders})\,, \qquad x^{1,2} \equiv \rho e^{\pm i\tau}$$

Write it as the undeformed metric plus a perturbation:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}$$

The CFT partition function is

$$\log Z = \log Z^{(0)} + \int \delta g_{\mu\nu}\langle T^{\mu\nu}\rangle + \frac{1}{2}\int \delta g_{\mu\nu}\delta g_{\rho\sigma}\langle T^{\mu\nu}T^{\rho\sigma}\rangle + (\text{higher orders})$$

- $\log Z^{(0)} \sim \text{Volume}\,(H^3)$ with cutoff $\rho > \epsilon$ has quadratic and logarithmic divergences.
- Our goal is to extract logarithmic divergences in the perturbation.

$$ds_4^2 = \frac{1}{\rho^2}\left[\frac{d\rho^2}{1+\rho^2} + \left(g_{ij} + 2K_{aij}x^a + Q_{abij}x^a x^b\right) dy^i dy^j\right] + (1+T\rho^2)d\tau^2 + 2U_i d\tau dy^i + (\text{higher orders})\,, \qquad x^{1,2} \equiv \rho e^{\pm i\tau}$$

$$\log Z = \log Z^{(0)} + \int \delta g_{\mu\nu}\langle T^{\mu\nu}\rangle + \frac{1}{2}\int \delta g_{\mu\nu}\delta g_{\rho\sigma}\langle T^{\mu\nu}T^{\rho\sigma}\rangle + (\text{higher orders})$$

- $\int \delta g_{\mu\nu}\langle T^{\mu\nu}\rangle$ produces $\log\epsilon$ for terms in $\delta g_{\mu\nu}$ quadratic in ρ. The coefficient of $\log\epsilon$ is schematically

$$-f_c(n)\int_\Sigma (T + Q^a{}_a) = -f_c(n)\int_\Sigma \left[C^{ab}{}_{ab} + \mathrm{Tr}K^2 + \frac{8}{3}U^2\right]$$

- $f_c(n)$ is determined by $\langle T^{\mu\nu}\rangle$ on the hyperboloid with $\beta = 2\pi n$, which can be computed holographically.
- Indeed it is related to $f_a(n)$ by

$$f_c(n) = \frac{n}{n-1}\left[a - f_a(n) - (n-1)f_a'(n)\right]\,.$$

$$
\begin{aligned}
ds_4^2 = \frac{1}{\rho^2}\left[\frac{d\rho^2}{1+\rho^2} + \left(g_{ij} + 2K_{aij}x^a + Q_{abij}x^a x^b\right) dy^i dy^j\right] \\
+ (1+T\rho^2)d\tau^2 + 2U_i d\tau dy^i + (\text{higher orders})\,, \qquad x^{1,2} \equiv \rho e^{\pm i\tau}
\end{aligned}
$$

$$
\log Z = \log Z^{(0)} + \int \delta g_{\mu\nu}\langle T^{\mu\nu}\rangle + \frac{1}{2}\int \delta g_{\mu\nu}\delta g_{\rho\sigma}\langle T^{\mu\nu}T^{\rho\sigma}\rangle + (\text{higher orders})
$$

- $\int \delta g_{\mu\nu}\delta g_{\rho\sigma}\langle T^{\mu\nu}T^{\rho\sigma}\rangle$ produces $\log\epsilon$ for terms in $\delta g_{\mu\nu}$ linear in ρ.
- They produces terms involving K^2 (and U^2).
- Computing $\langle T^{\mu\nu}T^{\rho\sigma}\rangle$ holographically gives the conjectured relation $f_b(n) = f_c(n)$ in the universal structure:

$$
S_n \sim \log\epsilon\left[\frac{f_a(n)}{2\pi}\int_\Sigma R_\Sigma + \frac{f_b(n)}{2\pi}\int_\Sigma \left(\mathrm{Tr}K^2 - \frac{1}{2}(\mathrm{Tr}K)^2\right) - \frac{f_c(n)}{2\pi}\int_\Sigma C^{ab}{}_{ab}\right]
$$

$$ds_4^2 = \frac{1}{\rho^2}\left[\frac{d\rho^2}{1+\rho^2} + \left(g_{ij} + 2K_{aij}x^a + Q_{abij}x^a x^b\right) dy^i dy^j\right] + (1+T\rho^2)d\tau^2 + 2U_i d\tau dy^i + (\text{higher orders})\,, \qquad x^{1,2} \equiv \rho e^{\pm i\tau}$$

$$\log Z = \log Z^{(0)} + \int \delta g_{\mu\nu}\langle T^{\mu\nu}\rangle + \frac{1}{2}\int \delta g_{\mu\nu}\delta g_{\rho\sigma}\langle T^{\mu\nu}T^{\rho\sigma}\rangle + (\text{higher orders})$$

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Quick "derivation": $\langle T^{ij}T^{kl}\rangle \propto \frac{1}{2}\left(g^{ik}g^{jl} + g^{il}g^{jk}\right) - \frac{1}{d}g^{ij}g^{kl}$.
Contracting with $K_{aij}K^a{}_{kl}$, we get $\tilde{f}_b(n)\left[\mathrm{Tr}K^2 - \frac{1}{4}(\mathrm{Tr}K)^2\right]$.
Combining this with $-f_c(n)\mathrm{Tr}K^2$, and requiring it to be $\propto\left[\mathrm{Tr}K^2 - \frac{1}{2}(\mathrm{Tr}K)^2\right]$, we find $\tilde{f}_b(n) = 2f_c(n) \Rightarrow f_b(n) = f_c(n)$.

Conclusion and Open Questions

1. There is a general formula that, evaluated on the conical defect C_1, gives the holographic EE in higher derivative gravity:

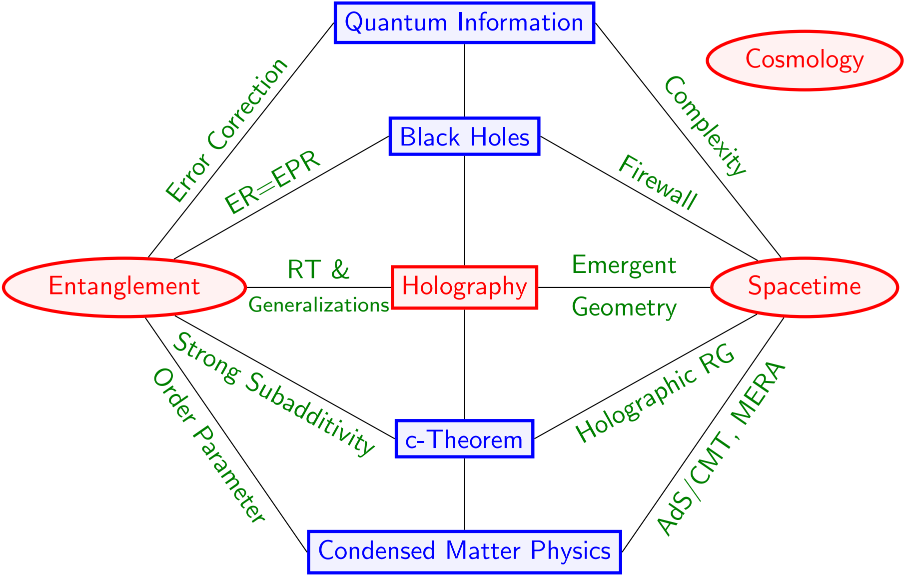
$$S_{EE} = 2\pi \int d^d y \sqrt{g} \left\{ \underbrace{\frac{\partial L}{\partial R_{z\bar{z}z\bar{z}}}}_{\text{Wald's formula}} + \underbrace{\sum_\alpha \left(\frac{\partial^2 L}{\partial R_{zizj} \partial R_{\bar{z}k\bar{z}l}} \right)_\alpha \frac{8 K_{zij} K_{\bar{z}kl}}{q_\alpha + 1}}_{\text{"Anomaly" from extrinsic curvature}} \right\}$$

2. Logarithmic terms in Renyi entropies for 4D CFTs have a universal structure that can be computed at least holographically:

$$S_n \sim \log \epsilon \left[\frac{f_a(n)}{2\pi} \int_\Sigma R_\Sigma + \frac{f_b(n)}{2\pi} \int_\Sigma \left(\mathrm{Tr} K^2 - \frac{1}{2} (\mathrm{Tr} K)^2 \right) - \frac{f_c(n)}{2\pi} \int_\Sigma C^{ab}{}_{ab} \right]$$

- Many open questions.
- How do we "enjoy" these results (in the big picture)?

Entanglement and Spacetime



Back Up Slides

Details on the Decomposition of the Riemann Tensor

$$ds^2 = e^{2A}\left[dzd\bar{z} + e^{2A}T(\bar{z}dz - zd\bar{z})^2\right] + \left(g_{ij} + 2K_{aij}x^a + Q_{abij}x^a x^b\right) dy^i dy^j \\ + 2ie^{2A}\left(U_i + V_{ai}x^a\right)(\bar{z}dz - zd\bar{z})\, dy^i + \cdots . \quad (1)$$

$$\begin{aligned}
R_{abcd} &= 12e^{4A}T\hat{\varepsilon}_{ab}\hat{\varepsilon}_{cd}\,, \\
R_{abci} &= 3e^{2A}\hat{\varepsilon}_{ab}V_{ci}\,, \\
R_{abij} &= 2e^{2A}\hat{\varepsilon}_{ab}(\partial_i U_j - \partial_j U_i) + g^{kl}(K_{ajk}K_{bil} - K_{aik}K_{bjl})\,, \\
R_{aibj} &= e^{2A}\left[\hat{\varepsilon}_{ab}(\partial_i U_j - \partial_j U_i) + 4\hat{g}_{ab}U_i U_j\right] + g^{kl}K_{ajk}K_{bil} - Q_{abij}\,, \\
R_{aijk} &= \partial_k K_{aij} - \gamma^l{}_{ik}K_{ajl} + 2\hat{\varepsilon}_{ab}\hat{g}^{bc}K_{cij}U_k - (j \leftrightarrow k)\,, \\
R_{ikjl} &= r_{ikjl} + e^{-2A}\hat{g}^{ab}(K_{ail}K_{bjk} - K_{aij}K_{bkl})\,,
\end{aligned}$$

Details on the Decomposition of the Riemann Tensor

$$ds^2 = e^{2A}\left[dzd\bar{z} + e^{2A}T(\bar{z}dz - zd\bar{z})^2\right] + \left(g_{ij} + 2K_{aij}x^a + Q_{abij}x^a x^b\right) dy^i dy^j$$
$$+ 2ie^{2A}\left(U_i + V_{ai}x^a\right)\left(\bar{z}dz - zd\bar{z}\right) dy^i + \cdots . \quad (2)$$

$$R_{abij} = \tilde{R}_{abij} + g^{kl}(K_{ajk}K_{bil} - K_{aik}K_{bjl}),$$
$$R_{aibj} = \tilde{R}_{aibj} + g^{kl}K_{ajk}K_{bil} - Q_{abij},$$
$$R_{ikjl} = r_{ikjl} + \hat{g}^{ab}(K_{ail}K_{bjk} - K_{aij}K_{bkl}),$$

$$\tilde{R}_{abij} \equiv 2e^{2A}\hat{\varepsilon}_{ab}(\partial_i U_j - \partial_j U_i),$$
$$\tilde{R}_{aibj} \equiv e^{2A}\left[\hat{\varepsilon}_{ab}(\partial_i U_j - \partial_j U_i) + 4\hat{g}_{ab}U_i U_j\right].$$

Details on the Squashed Cone

$$ds^2 = d\tilde{\rho}^2 + \tilde{\rho}^2 \left[1 + \tilde{\rho}^2 \mathcal{O}\left(1, \tilde{\rho}^2, \tilde{\rho}^n e^{\pm in\tilde{\tau}}\right)\right] d\tilde{\tau}^2$$
$$+ \left[g_{ij} + \mathcal{O}\left(\tilde{\rho}^2, \tilde{\rho}^n e^{\pm in\tilde{\tau}}\right)\right] dy^i dy^j + \tilde{\rho}^2 \mathcal{O}\left(1, \tilde{\rho}^2, \tilde{\rho}^n e^{\pm in\tilde{\tau}}\right) d\tilde{\tau} dy^i \,. \quad (3)$$

$$\mathcal{O}\left(1, \tilde{\rho}^2, \tilde{\rho}^n e^{\pm in\tilde{\tau}}\right) \equiv \sum_{k=-\infty}^{\infty} \left(\sum_{m=0}^{\infty} \tilde{c}_{km} \tilde{\rho}^{2m}\right) \tilde{\rho}^{|k|n} e^{\pm ikn\tilde{\tau}} \,,$$

$$\rho \equiv \left(\frac{\tilde{\rho}}{n}\right)^n \,, \quad \tau \equiv n\tilde{\tau} \,,$$

$$ds^2 = \rho^{-2\epsilon} \left\{d\rho^2 + \rho^2 \left[1 + \rho^{2-2\epsilon} \mathcal{O}\left(1, \rho^{2-2\epsilon}, \rho e^{\pm i\tau}\right)\right] d\tau^2\right\}$$
$$+ \left[g_{ij} + \mathcal{O}\left(\rho^{2-2\epsilon}, \rho e^{\pm i\tau}\right)\right] dy^i dy^j + \rho^{2-2\epsilon} \mathcal{O}\left(1, \rho^{2-2\epsilon}, \rho e^{\pm i\tau}\right) d\tau dy^i \,. \quad (4)$$

$$\mathcal{O}\left(1, \rho^{2-2\epsilon}, \rho e^{\pm i\tau}\right) \equiv \sum_{k=-\infty}^{\infty} \left(\sum_{m=0}^{\infty} c_{km} \rho^{(2-2\epsilon)m}\right) \rho^{|k|} e^{\pm ik\tau} \,,$$

Caveat

Caveat: in the prescription for q_α, there might be an ambiguity about how to count $Q_{z\bar{z}ij}$. We made a particular assumption about the analytic continuation of the Z_n-symmetric metric to non-integer n:

$$G_{ij} = g_{ij} + 2K_{zij}z + 2K_{\bar{z}ij}\bar{z} + Q_{zzij}z^2 + 2Q_{z\bar{z}ij}(z\bar{z})^{1/n} + \cdots$$

In parent space: w^n $\bar{w}^n$ w^{2n} $w\bar{w}$

But can there be a term $2\tilde{Q}_{z\bar{z}ij}z\bar{z} \sim w^n\bar{w}^n$? Answer should come from EOM.

Details on Lovelock gravity

$$S^{(2p)} = \int d^D x \sqrt{G} L^{(2p)}$$

$$L^{(2p)} = \frac{1}{2^p} \delta^{\mu_1 \rho_1 \mu_2 \rho_2 \cdots \mu_p \rho_p}_{\nu_1 \sigma_1 \nu_2 \sigma_2 \cdots \nu_p \sigma_p} R_{\mu_1 \rho_1}{}^{\nu_1 \sigma_1} R_{\mu_2 \rho_2}{}^{\nu_2 \sigma_2} \cdots R_{\mu_p \rho_p}{}^{\nu_p \sigma_p}$$

$$E^{(2p)\mu\nu} = \frac{1}{\sqrt{G}} \frac{\delta S^{(2p)}}{\delta G_{\mu\nu}} = \frac{1}{2} G^{\mu\nu} L^{(2p)} - L_4^{(2p)\mu\rho_1\nu_1\sigma_1} R^{\nu}{}_{\rho_1 \nu_1 \sigma_1}$$

$$E^{(2p)\mu}{}_{\nu} = \frac{1}{2^{p+1}} \delta^{\mu \mu_1 \rho_1 \mu_2 \rho_2 \cdots \mu_p \rho_p}_{\nu \nu_1 \sigma_1 \nu_2 \sigma_2 \cdots \nu_p \sigma_p} R_{\mu_1 \rho_1}{}^{\nu_1 \sigma_1} R_{\mu_2 \rho_2}{}^{\nu_2 \sigma_2} \cdots R_{\mu_p \rho_p}{}^{\nu_p \sigma_p}$$

$$L_4^{\mu\rho\nu\sigma} = \frac{1}{\sqrt{G}} \frac{\delta S}{\delta R_{\mu\rho\nu\sigma}}$$

Details on Minimization: f(R) Gravity

$$S_{EE} = -4\pi \int d^d y \sqrt{g} \frac{\partial L}{\partial R}$$

Claim: Minimizing this gives the location of C_n at $n = 1$

- Proof 1: Transform to Einstein gravity + scalar.
- Proof 2: Cosmic brane method:

$$\begin{aligned} S_{\text{total}} &= \lambda \int d^D x \sqrt{G} R^p - 4\pi p \lambda \epsilon \int d^d y \sqrt{g} R^{p-1} \\ &= \lambda \int d^D x \sqrt{G} R^p - 4\pi p \lambda \epsilon \int d^D x \sqrt{g} R^{p-1} \delta(x^1, x^2) \end{aligned}$$

Solve the most singular terms in EOM:

$$\frac{\delta S_{\text{total}}}{\delta G_{\mu\nu}} \sim p \nabla_a \nabla_b R^{p-1} - 4\pi p(p-1) R^{p-2} \nabla_a \nabla_b \delta(x^1, x^2) + \cdots$$

Therefore $R \sim -2\nabla^2 A$ needs to produce $4\pi\delta(x^1, x^2) \Rightarrow A = -\epsilon \log \rho$.

Details on Minimization: General 4-Derivative Gravity

$$L = \lambda_1 R^2 + \lambda_2 R_{\mu\nu} R^{\mu\nu} + \lambda_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$$

$$S_{EE} = -4\pi \int d^d y \sqrt{g} \left\{ 2\lambda_1 R + \lambda_2 \left(R^a{}_a - \frac{1}{2} K_a K^a \right) + 2\lambda_3 \left(R^{ab}{}_{ab} - K_{aij} K^{aij} \right) \right\}$$

- Can also show minimizing this gives the location of C_n at $n = 1$.
- Use the cosmic brane method.
- Extrinsic curvature terms show up to compensate

$$\frac{\delta S}{\delta G_{\mu\nu}} \supset \nabla_z \nabla_z R_{\bar{z}i\bar{z}j} \supset (\nabla_z \nabla_z \nabla_{\bar{z}} A) K_{\bar{z}ij}$$

in EOM by providing e.g. $\nabla_z \delta(x^1, x^2) K_{\bar{z}ij}$.

Details on Minimization: General Lovelock Gravity

$$L_{2p}(R) = \frac{1}{2^p}\delta^{\mu_1\rho_1\mu_2\rho_2\cdots\mu_p\rho_p}_{\nu_1\sigma_1\nu_2\sigma_2\cdots\nu_p\sigma_p} R_{\mu_1\rho_1}{}^{\nu_1\sigma_1} R_{\mu_2\rho_2}{}^{\nu_2\sigma_2}\cdots R_{\mu_p\rho_p}{}^{\nu_p\sigma_p}$$

$$S_{EE} = -4\pi p \int d^d y \sqrt{g} L_{2p-2}(r)$$

Cosmic brane method

- Lovelock is simple because EOM is 2-derivative, no ∇R.
- Simply match coefficients of $\delta(x^1, x^2)$ to linear order in ϵ.
- "Explains" why S_{EE} depends only on d-dim'l intrinsic curvature r.

Boundary condition method (generalizing [Bhattacharyya Sharma Sinha 1308.5748])

The zz component of "Einstein equation" is potentially divergent:

$$E^{\bar{z}}{}_z = \frac{1}{2^{p+1}}\delta^{\bar{z}\mu_1\rho_1\mu_2\rho_2\cdots\mu_p\rho_p}_{z\nu_1\sigma_1\nu_2\sigma_2\cdots\nu_p\sigma_p} R_{\mu_1\rho_1}{}^{\nu_1\sigma_1} R_{\mu_2\rho_2}{}^{\nu_2\sigma_2}\cdots R_{\mu_p\rho_p}{}^{\nu_p\sigma_p}$$

$$\sim \frac{\epsilon}{z} K_{zi}{}^{j}\delta^{ii_1k_1i_2k_2\cdots i_{p-1}k_{p-1}}_{jj_1l_1j_2l_2\cdots j_{p-1}l_{p-1}} R_{i_1k_1}{}^{j_1l_1} R_{i_2k_2}{}^{j_2l_2}\cdots R_{i_{p-1}k_{p-1}}{}^{j_{p-1}l_{p-1}}$$

Precisely the equation $\frac{\delta S_{EE}}{\delta g_{ij}} K_{zij} = 0$ from minimizing S_{EE}!

Details on One-Loop Bulk Correction

- Given by the functional determinant of the operator describing quadratic fluctuations of all the bulk fields.
- For AdS_3/Γ there is an elegant expression. [Giombi, Maloney & Yin 0804.1773; Yin 0710.2129]

For metric fluctuations:

$$\log Z|_{\text{one-loop}} = -\sum_{\gamma\in\mathcal{P}}\sum_{m=2}^{\infty}\log|1-q_\gamma^m|$$

- ◇ $\mathcal{P}$ is a set of representatives of the primitive conjugacy classes of Γ.
- ◇ q_γ is defined by writing the two eigenvalues of $\gamma \in \Gamma \subset PSL(2,\mathbb{C})$ as $q_\gamma^{\pm 1/2}$ with $|q_\gamma| < 1$.
- ◇ Similar expressions exist for other bulk fields.

Nice feature 2: at integer n the sum can be done explicitly in terms of rational functions of n:

$$S_n|_{\text{one-loop}} = -\frac{n}{n-1}\sum_{k=1}^{n-1}\left[\frac{\csc^8}{256n^8}x^4 + \frac{(n^2-1)\csc^8+\csc^{10}}{128n^{10}}x^5 + \mathcal{O}(x^6)\right]$$
$$= \frac{(n+1)(n^2+11)(3n^4+10n^2+227)}{3628800n^7}x^4 + \mathcal{O}(x^5)$$

$$\text{where} \quad \csc \equiv \csc\left(\frac{\pi k}{n}\right)$$

(4) Analytically continue the one-loop result to $n \to 1$:

$$S|_{\text{one-loop}} = -\left(\frac{x^4}{630} + \frac{2x^5}{693} + \frac{15x^6}{4004} + \frac{x^7}{234} + \frac{167x^8}{36936} + \mathcal{O}(x^9)\right)$$

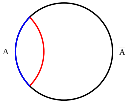
Exactly agrees with known results at leading order:

$$S = -\mathcal{N}\left(\frac{x}{4}\right)^{2h}\frac{\sqrt{\pi}}{4}\frac{\Gamma(2h+1)}{\Gamma\left(2h+\frac{3}{2}\right)} + \cdots$$ [Calabrese, Cardy & Tonni '11]

Details on One-Loop Corrections in the Torus Case

Nice feature: only "single-letter" words $\{L_i, L_i^{-1}\}$ contribute to the leading order in the low / high T limit.

Thermal AdS:



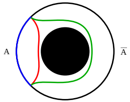
$$S_n|_{\text{one-loop}} = -\frac{1}{n-1}\left[\frac{2\sin^4\left(\frac{\pi L}{R}\right)}{n^3\sin^4\left(\frac{\pi L}{nR}\right)} - 2n\right] e^{-\frac{4\pi}{TR}} + \mathcal{O}\left(e^{-\frac{6\pi}{TR}}\right)$$

$$S|_{\text{one-loop}} = \left[-\frac{8\pi L}{R}\cot\left(\frac{\pi L}{R}\right) + 8\right] e^{-\frac{4\pi}{TR}} + \mathcal{O}\left(e^{-\frac{6\pi}{TR}}\right)$$

$$\Rightarrow \quad S_A - S_{\bar{A}} = -8\pi\cot\left(\frac{\pi L_A}{R}\right) e^{-\frac{4\pi}{TR}} + \mathcal{O}\left(e^{-\frac{6\pi}{TR}}\right)$$

Agrees (morally) with a free field calculation in [Herzog & Spillane 1209.6368].

BTZ:



$$S_n|_{\text{one-loop}} = -\frac{1}{n-1}\left[\frac{2\sinh^4(\pi TL)}{n^3\sinh^4\left(\frac{\pi TL}{n}\right)} - 2n\right] e^{-4\pi TR} + \mathcal{O}\left(e^{-6\pi TR}\right)$$

$$S|_{\text{one-loop}} = \left[-8\pi TL\coth(\pi TL) + 8\right] e^{-4\pi TR} + \mathcal{O}\left(e^{-6\pi TR}\right)$$

Where to Evaluate the Entropy Formula?

Should evaluate it at the conical defect C_n as $n \longrightarrow 1$

- C_1 is well-defined in principle but hard to find using its definition.
- Can it be found by minimizing some functional?
- In the cosmic brane method, we ask: What is S_B that creates a conical defect in higher derivative gravity, to linear order in ϵ?

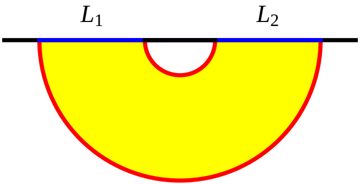
In particular, can this simply be S_{EE} that we saw?

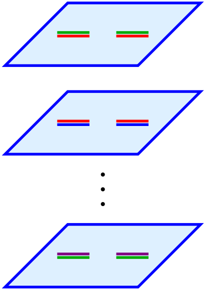
Yes, at least for three classes of examples:

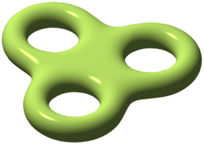
- $f(R)$ gravity
- General 4-derivative gravity
- Lovelock gravity

[XD 1310.5713] [Bhattacharyya, Sharma & Sinha 1308.5748]

One-Loop Corrections to Ryu-Takayanagi







[Barrella, XD, Hartnoll & Martin 1306.4682]:

$$I = \frac{x^4}{630} + \frac{2x^5}{693} + \frac{15x^6}{4004} + \frac{x^7}{234} + \frac{167x^8}{36936} + \mathcal{O}(x^9)$$

Exactly agrees with CFT predictions:
[Calabrese, Cardy & Tonni '11; Chen & Zhang 1309.5453]

Can also generalize to finite temperature: