A Proof of the Covariant Entropy Bound

Joint work with H. Casini, Z. Fisher, and J. Maldacena,
arXiv:1404.5635 and 1406.4545

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The World as a Hologram

- The Covariant Entropy Bound is a relation between information and geometry. RB 1999
- Motivated by holographic principle.
  Bekenstein 1972; Hawking 1974
  't Hooft 1993; Susskind 1995; Susskind and Fischler 1998
- Conjectured to hold in arbitrary spacetimes, including cosmology.
- The entropy on a light-sheet is bounded by the difference between its initial and final area in Planck units.
- If correct, origin must lie in quantum gravity.

A Proof of the Covariant Entropy Bound

- In this talk I will present a proof, in the regime where gravity is weak ($G\hbar \to 0$).
- Though this regime is limited, the proof is interesting.
- No need to assume any relation between the entropy and energy of quantum states, beyond what quantum field theory already supplies.
- This suggests that quantum gravity determines not only classical gravity, but also nongravitational physics, as a unified theory should.

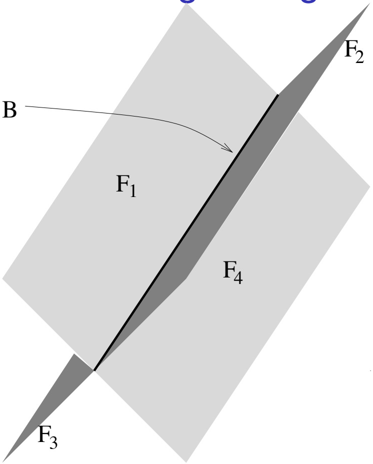
Covariant Entropy Bound

Entropy ΔS

Modular Energy ΔK

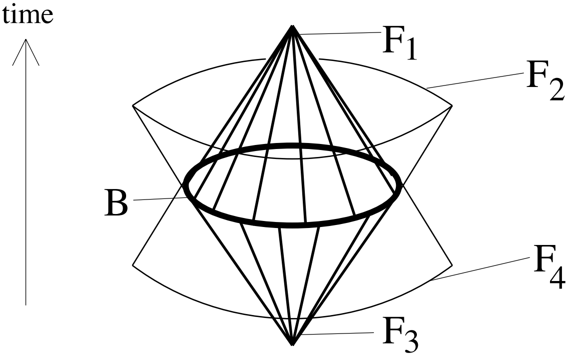
Area Loss ΔA

Surface-orthogonal light-rays



- Any 2D spatial surface *B* bounds four (2+1D) null hypersurfaces
- Each is generated by a congruence of null geodesics ("light-rays") $\perp$ *B*

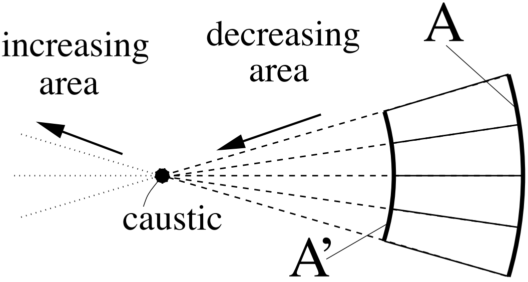
Light-sheets



Out of the 4 orthogonal directions, usually at least 2 will initially be nonexpanding.

The corresponding null hypersurfaces are called light-sheets.

The Nonexpansion Condition



$$\theta = \frac{d\mathcal{A}/d\lambda}{\mathcal{A}}$$

Demand

$\theta \leq 0 \quad \leftrightarrow \quad$ nonexpansion

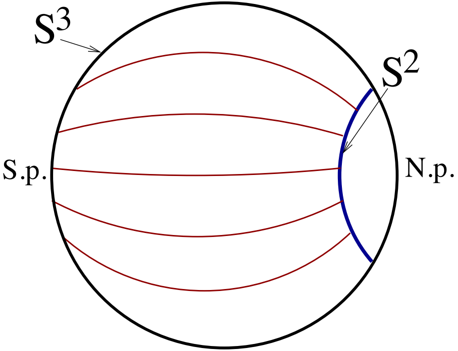
everywhere on the light-sheet.

Covariant Entropy Bound

In an arbitrary spacetime, choose an arbitrary two-dimensional surface B of area A. Pick any light-sheet of B.
Then $S \leq A/4G\hbar$, where S is the entropy on the light-sheet.

RB 1999

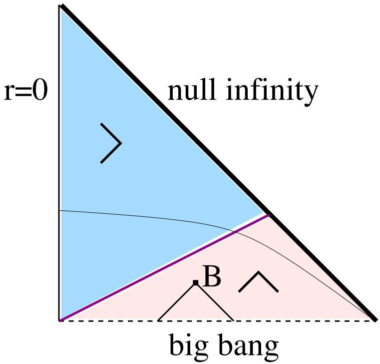
Example: Closed Universe



(a)

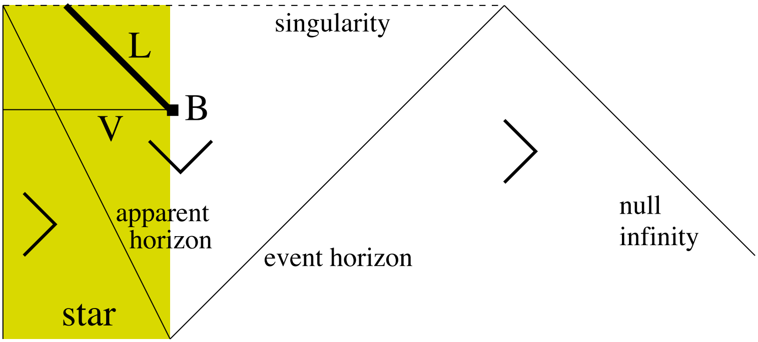
- S(volume of most of $\mathbf{S}^3$) $\gg A(\mathbf{S}^2)$
- The light-sheets are directed towards the "small" interior, avoiding an obvious contradiction.

Example: Flat FRW universe



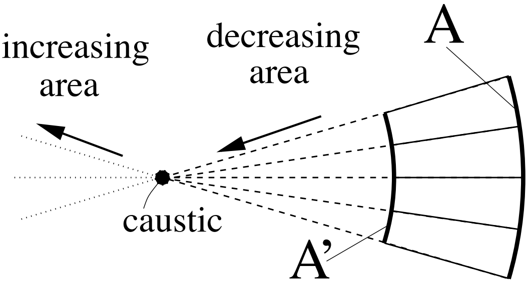
- Sufficiently large spheres at fixed time t are anti-trapped
- Only past-directed light-sheets are allowed
- The entropy on these light-sheets grows only like R^2

Example: Collapsing star



- At late times the surface of the star is trapped
- Only future-directed light-sheets exist
- They do not contain all of the star

Generalized Covariant Entropy Bound

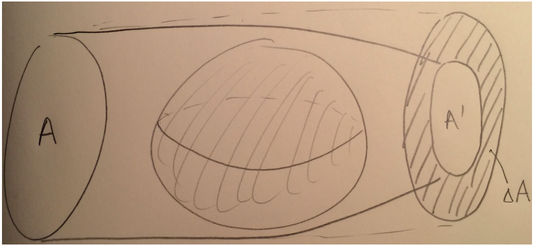


If the light-sheet is terminated at finite cross-sectional area A', then the covariant bound can be strengthened:

$$S \leq \frac{A - A'}{4G\hbar}$$

Flanagan, Marolf & Wald, 1999

Generalized Covariant Entropy Bound



$$S \leq \frac{\Delta A}{4G\hbar}$$

For a given matter system, the tightest bound is obtained by choosing a nearby surface with initially vanishing expansion.

Bending of light implies

$$A - A' \equiv \Delta A \propto G\hbar \,.$$

Hence, the bound remains nontrivial in the weak-gravity regime ($G\hbar \to 0$).

RB 2003

Covariant Entropy Bound

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Modular Energy ΔK

Area Loss ΔA

How is the entropy defined?

- In cosmology, and for well-isolated systems: usual, "intuitive" entropy. But more generally?

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- In cosmology, and for well-isolated systems: usual, “intuitive” entropy. But more generally?
- Quantum systems are not sharply localized. Under what conditions can we consider a matter system to “fit” on L?
- The vacuum, restricted to L, contributes a divergent entropy. What is the justification for ignoring this piece?

In the $G\hbar \to 0$ limit, a sharp definition of S is possible.

Vacuum-subtracted Entropy

Consider an arbitrary state ρ_{global}. In the absence of gravity, $G = 0$, the geometry is independent of the state. We can restrict both ρ_{global} and the vacuum $|0\rangle$ to a subregion V:

$$\begin{aligned} \rho &\equiv \text{tr}_{-V}\, \rho_{\text{global}} \\ \rho_0 &\equiv \text{tr}_{-V}\, |0\rangle\langle 0| \end{aligned}$$

The von Neumann entropy of each reduced state diverges like A/ϵ^2, where A is the boundary area of V, and ϵ is a cutoff. However, the difference is finite as $\epsilon \to 0$:

$$\Delta S \equiv S(\rho) - S(\rho_0)\ .$$

Marolf, Minic & Ross 2003, Casini 2008

Properties of Vacuum-subtracted Entropy

- For excitations that are well localized to the interior of V, one recovers the “intuitive entropy”, $\Delta S \approx S(\rho_{\text{global}})$

Properties of Vacuum-subtracted Entropy

- For excitations that are well localized to the interior of V, one recovers the "intuitive entropy", $\Delta S \approx S(\rho_{\text{global}})$
- For an incoherent superposition of n light species, $S(\rho_{\text{global}})$ diverges logarithmically with n.
  But ΔS saturates. $\rightarrow$ No Species Problem
- Physically, an observer with access only to V cannot discriminate an arbitrary number of species, due to thermal effects.

Covariant Entropy Bound

Entropy ΔS

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Relative Entropy

Given any two states, the (asymmetric!) *relative entropy*

$$S(\rho|\rho_0) = -\mathrm{tr}\,\rho \log \rho_0 - S(\rho)$$

satisfies positivity and monotonicity: under restriction of ρ and ρ_0 to a subalgebra (e.g., a subset of V), the relative entropy cannot increase.

Lindblad 1975

Modular Hamiltonian

Definition: Let ρ_0 be the vacuum state, restricted to some region V. Then the *modular Hamiltonian*, K, is defined up to a constant by

$$\rho_0 \equiv \frac{e^{-K}}{\mathrm{tr}\, e^{-K}} \ .$$

The *modular energy* is defined as

$$\Delta K \equiv \mathrm{tr}\, K\rho - \mathrm{tr}\, K\rho_0$$

A Central Result

Positivity of the relative entropy implies immediately that

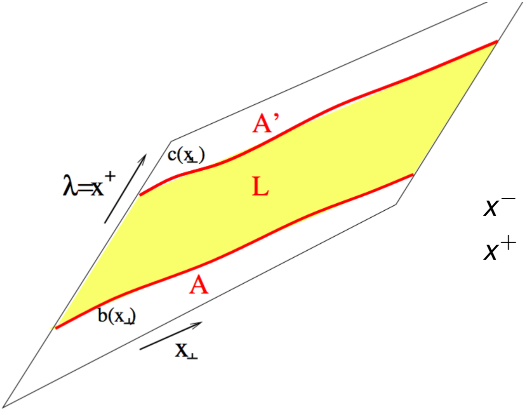
$$\Delta S \leq \Delta K \ .$$

To complete the proof, we must compute ΔK and show that

$$\Delta K \leq \frac{\Delta A}{4G\hbar} \ .$$

Light-sheet Modular Hamiltonian

In finite spatial volumes, the modular Hamiltonian K is nonlocal.
But we consider a portion of a null plane in Minkowski:



$$x^- \equiv t - x = 0\,;$$
$$x^+ \equiv t + x\,; 0 < x^+ < 1\,.$$

In this case, K simplifies dramatically.

Free Case

- The vacuum on the null plane factorizes over its null generators.
- The vacuum on each generator is invariant under a special conformal symmetry. Wall (2011)

Thus, we may obtain the modular Hamiltonian by application of an inversion, $x^+ \to 1/x^+$, to the (known) Rindler Hamiltonian on $x^+ \in (1, \infty)$. We find

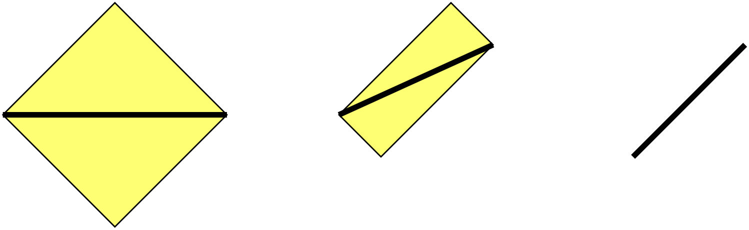
$$K = \frac{2\pi}{\hbar} \int d^{d-2}y \int_0^1 dx^+ \, g(x^+) \, T_{++}$$

with

$$g(x^+) = x^+(1 - x^+) \, .$$

Interacting Case

In this case, it is not possible to define ΔS and K directly on the light-sheet. Instead, consider the null limit of a spatial slab:



(a) (b) (c)

Interacting Case

We cannot compute ΔK on the spatial slab.

However, it is possible to constrain the form of ΔS by analytically continuing the Rényi entropies,

$$S_n = (1-n)^{-1} \log \mathrm{tr} \rho^n ,$$

to $n = 1$.

Interacting Case

The Rényi entropies can be computed using the replica trick,

Calabrese and Cardy (2009)

as the expectation value of a pair of defect operators inserted at the boundaries of the slab. In the null limit, this becomes a null OPE to which only operators of twist d-2 contribute. The only such operator in the interacting case is the stress tensor, and it can contribute only in one copy of the field theory.

This implies

$$\Delta S = \frac{2\pi}{\hbar} \int d^{d-2}y \int_0^1 dx^+ \; g(x^+) \; T_{++} \; .$$

Interacting Case

Because ΔS is the expectation value of a linear operator, it follows that

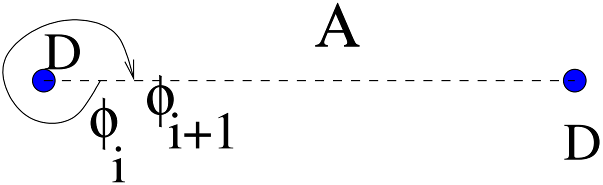
$$\Delta S = \Delta K$$

for all states.

Blanco, Casini, Hung, and Myers 2013

This is possible because the operator algebra is infinite-dimensional; yet any given operator is eliminated from the algebra in the null limit.

Once More, With Feeling



The Rényi entropies for an interval *A* involve the two point function of defect operators *D* inserted at the endpoints of the interval.

An operator in the i^{th} CFT becomes an operator in the $(i+1)^{th}$ CFT when we go around the defect.

Take the limit in which the interval becomes null.

Once More, With Feeling

Light-like OPE: Take $x^2 \to 0$ with $x^+ \equiv x^0 + x^1$ held fixed.
The expansion of two scalar operators has the form

$$O(x)O(0) \sim \sum_k |x|^{-2\tau_O + \tau_k} (x^+)^{s_k} O_{k,s_k}.$$

Here $s_k =$ spin, $\Delta_k =$ scaling dimension.

We see that the twist $\tau_k \equiv \Delta_k - s_k$ governs the approach to the light-like limit.

For finite x^+, we sum over all contributions with a given twist.

Once More, With Feeling

In $d > 2$ we consider instead a spatial slab, with codimension 2 defect operators inserted.

Then we take the limit in which the slab becomes null.

Once More, With Feeling

Spacelike OPE of defect operators:

$$D(x)D(0) \sim \exp\left\{\int d^{d-2}y\left[\sum_k \frac{1}{|x|^{d-2-\Delta_k}} O_k(x=0,y)\right]\right\}$$

where y denotes the transverse dimensions and O_k denotes local operators on the defect at $x=0$. Thus the expansion is local in y.

Once More, With Feeling

Light-like defect OPE:

$$D(x)D(0) \sim \exp\left\{\int d^{d-2}y \left[\sum_k |x|^{-(d-2)+\tau_k}(x^+)^{s_k} O_{k,s_k}\right]\right\} .$$

Once More, With Feeling

The leading term in the OPE is given by the identity operator and contributes a factor of $A_y/|x|^{d-2}$ in the exponent (with a coefficient that depends on n), where A_y is the transverse area.

This is the expected form of $\mathrm{tr}\rho_0^n = e^{-(n-1)S_n}$, the vacuum Rényi entropies for the slab. In the vacuum case, all other operators have vanishing expectation values.

This contribution cancels when we compute the difference ΔS, so we need not consider it further.

Once More, With Feeling

In an interacting theory, all operators with spin greater than 2 are expected to have twist strictly larger than $d-2$. The twist is expected to increase as the spin increases.

Komargodski and Zhiboedov 2013

The only operator with spin 2 and twist $d-2$ is the stress tensor.

We argue that operators with lower spin have twist $> \frac{d-2}{2}$ and must appear in pairs for symmetry reasons $\rightarrow$ no contribution.

Once More, With Feeling

The generic form of the operators in the expansion is

$$O = O_1 O_2 \cdots O_n \,,$$

where O_k is an operator on the k^{th} copy of the original CFT.

For $d > 2$, the leading twist operators have only one factor which is not the identity (the stress tensor). Performing the replica trick, they contribute to the entropy proportionally to an operator in the original CFT:

$$S_{\mathrm{single}} = \langle O_S \rangle \,.$$

Such contributions are linear in the density matrix, and therefore do not give rise to a non-zero value of $\Delta K - \Delta S$.

K is the only operator localized to the region whose expectation value coincides with ΔS to linear order for any deviation from the vacuum state.

Blanco, Casini, Hung, and Myers 2013

Once More, With Feeling

All of the descendants of T_{++} contribute as well, so the OPE becomes a Taylor expansion around $x^+ = 0$:

$$D_n(x)D_n(0) \sim \exp\left\{-(n-1)2\pi \int d^{d-2}y \int_0^1 dx^+\, g_n(x^+)\, T_{++}]\right\} .$$

Once More, With Feeling

The vacuum-subtracted von Neumann entropy is then given by analytic continuation:

$$\begin{aligned}\Delta S &= \lim_{n\to 1}(1-n)^{-1}\log\langle D_n(x)D_n(0)\rangle \\ &= 2\pi\int d^{d-2}y\int_0^1 dx^+ g(x^+)\,T_{++}(x^-=0,x^+,y) \\ &= \Delta K\,.\end{aligned}$$

The function g is as yet undetermined.

Interacting Case

We thus have

$$\Delta K = \frac{2\pi}{\hbar} \int d^{d-2}y \int_0^1 dx^+ \, g(x^+) \; T_{++} \; .$$

Known properties of the modular Hamiltonian of a region and its complement further constrain the form of $g(x^+)$:

$g(0) = 0$, $g'(0) = 1$, $g(x^+) = g(1 - x^+)$, and $|g'| \leq 1$.

I will now show that these properties imply

$$\Delta K \leq \Delta A / 4G\hbar \; ,$$

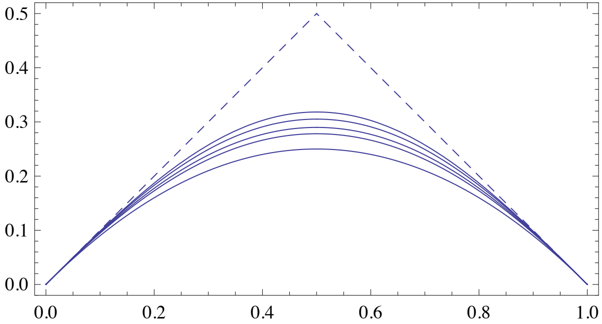
which completes the proof.

Interacting Case

For interacting theories with a gravity dual we are able to compute $g(x^+)$ from the area of extremal surfaces:

Ryu and Takayanagi (2006)

Hubeny, Rangamani and Takayanagi (2007)



For $d > 2$, we find

$$g(x^+) \neq x^+(1 - x^+) \ .$$

Covariant Entropy Bound

Entropy ΔS

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Area Loss in the Weak Gravity Limit

Integrating the Raychaudhuri equation twice, one finds

$$\Delta A = -\int_0^1 dx^+ \theta(x^+) = -\theta_0 + 8\pi G \int_0^1 dx^+ (1 - x^+) T_{++} \ .$$

at leading order in G.

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$$\Delta K = \frac{2\pi}{\hbar} \int_0^1 dx^+ \ g(x^+) \ T_{++} \ .$$

Since $\theta_0 \leq 0$ and $g(x^+) \leq (1 - x_+)$, we have $\Delta K \leq \Delta A / 4G\hbar$

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if we assume the Null Energy Condition, $T_{++} \geq 0$.

Violations of the Null Energy Condition

- It is easy to find quantum states for which $T_{++} < 0$.
- Explicit examples can be found for which $\Delta S > \Delta A / 4G\hbar$, if $\theta_0 = 0$.
- Perhaps the Covariant Entropy Bound must be modified if the NEC is violated?
- E.g., evaporating black holes

Lowe 1999

Strominger and Thompson 2003

- Surprisingly, we can prove $S \leq (A - A')/4$ without assuming the NEC.

Negative Energy Constrains θ_0

- If the null energy condition holds, $\theta_0 = 0$ is the “toughest” choice for testing the Entropy Bound.
- However, if the NEC is violated, then $\theta_0 = 0$ does not guarantee that the nonexpansion condition holds everywhere.
- To have a valid light-sheet, we must require that

$$0 \geq \theta(x^+) = \theta_0 + 8\pi G \int_{x^+}^{1} d\hat{x}^+ \, T_{++}(\hat{x}^+) \, ,$$

  holds for all $x^+ \in [0, 1]$.
- This can be accomplished in any state.
- But the light-sheet may have to contract initially:

$$\theta_0 \sim O(G\hbar) < 0 \, .$$

Proof of $\Delta K \leq \Delta A / 4G\hbar$

Let $F(x^+) = x^+ + g(x^+)$. The properties of g imply $F' \geq 0$, $F(0) = 0$, $F(1) = 1$.

By nonexpansion, we have $0 \geq \int_0^1 F' \, \theta \, dx^+$, and thus

$$\theta_0 \leq 8\pi G \int dx^+ [1 - F(x^+)] T_{++} \ . \tag{1}$$

For the area loss, we found

$$\Delta A = -\int_0^1 dx^+ \theta(x^+) = -\theta_0 + 8\pi G \int_0^1 dx^+ (1 - x^+) T_{++} \ . \tag{2}$$

Combining both equations, we obtain

$$\frac{\Delta A}{4G\hbar} \geq \frac{2\pi}{\hbar} \int_0^1 dx^+ \, g(x^+) \, T_{++} = \Delta K \ . \tag{3}$$

Monotonicity

- In all cases where we can compute g explicitly, we find that it is concave:
$$g'' \leq 0$$
- This property implies the stronger result of monotonicity:
- As the size of the null interval is increased, $\Delta S - \Delta A/4G\hbar$ is nondecreasing.
- No general proof yet.

Covariant Bound vs. Generalized Second Law

- The Covariant Entropy Bound applies to any null hypersurface with $\theta \leq 0$ everywhere.
- It constrains the vacuum subtracted entropy on a finite null slab.
- The GSL applies only to causal horizons, but does not require $\theta \leq 0$.
- It constrains the entropy difference between two nested semi-infinite null regions.
- Limited proofs exist for both, but is there a more direct relation?