Problem Set 9

Problem 1

For a Dirac spinor field, show that

$$\langle \psi_\alpha(x)\psi_\beta(y)\rangle = 0 \ .$$

Problem 2: Interacting field theory

Suppose we have a field theory in D dimensions (including time), with vector, scalar, and spinor fields whose kinetic terms take the standard form (schematically)

$$S = \int d^D x \{(\partial\phi)^2 + \bar{\psi}\gamma\partial\psi + (\partial A)^2\}$$

a) With our usual choice of units so that $\hbar = c = 1$, all inverse lengths, inverse times, and energies have the same units (which we can take to be eV, i.e. some unit of energy). In these units, the action is dimensionless. What units do the fields ϕ, ψ, and A have (e.g. E, E^2, etc...)?

b) Let's focus for now on scalar field theory, and assume for simplicity the massless case. Suppose we have interaction terms in the action

$$S_{int} = \sum_{n>2} \int d^D x \lambda_n \phi^n \ . \tag{1}$$

What are the units of λ?

c) Now, if these terms can be treated as a perturbations to the free theory (i.e. they correspond to sufficiently weak interactions), it makes sense to expand the results for some physical observable (e.g. the probability for some scattering) in powers of the coefficients λ_n.

$$P = P_0(1 + \lambda_n \Delta_n + \cdots) \ .$$

Here, the quantities Δ will be some functions of the experimental input (e.g. the energy and/or momenta of the particles involved). Based on dimensional analysis, if we scale all particle energies in the problem by a factor of ϵ, how do the quantities Δ_n scale?

d) From the previous part, you should see that the contributions of some terms become less important for small energies, while the contributions of some terms become more important for small energies (or remain the same)? For $D = 4$, what terms in the action (1) give contributions that do not become less and less important for low energies (these terms are known as RELEVANT interactions)?

e) Going back to the case with scalar, vector, and (complex) spinor field, and using a similar analysis, write all possible *relevant* Lorentz-invariant interaction terms (i.e. terms involving three or more of the fields).

Problem 3: Neutrinos

In the Standard Model of Particle Physics, neutrinos were originally assumed to be massless spin-half particles. Since they only interact via the weak interaction, and weak interactions don't have a parity symmetry, it is not necessary to use a full Dirac spinor field for a neutrino; the two-component spinor fields (called Weyl spinors) are sufficient. Alternately, we can work with a Dirac spinor field, but define:

$$\psi_L = \frac{1}{2}(1-\gamma^5)\psi \qquad\qquad \psi_R = \frac{1}{2}(1+\gamma^5)\psi \ .$$

a) If we define $\psi = \begin{pmatrix} \eta_a \\ \chi_a \end{pmatrix}$, show that ψ_L and ψ_R each contain only two independent components (η and χ respectively), and argue that ψ_L and ψ_R do not mix under Lorentz transformation (excluding parity). Thus, it is possible to build a Lorentz-invariant (but not parity-invariant) action using ψ_L alone.

b) Show that $\bar{\psi}\gamma^\mu\partial_\mu\psi$ may be written as a sum of separate actions for ψ_L and ψ_R, so that $\bar{\psi}_L\gamma^\mu\partial_\mu\psi_L$ could be used as an action in a theory with ψ_L alone. This is how neutrinos appear in the Standard Model.

c) Write out $\bar{\psi}\psi$ in terms of ψ_L and ψ_R. Show that $\bar{\psi}_L\psi_L$ vanishes. Thus, we cannot have a mass term $\bar{\psi}_L\psi_L$ for a field ψ_L alone. Since we now have evidence that neutrinos are massive, one possibility is that the standard model also contains a right-handed neutrino field ψ_R in addition to the ψ_L that was used originally.

d) Since neutrinos are not charged fields, another possibility is that they have a mass term which doesn't have a $\psi \to e^{i\theta}\psi$ symmetry. Such a term looks like:

$$S = \int d^4x \left\{ \frac{im}{2}(\psi_L^T C \psi_L - \psi_L^\dagger C \psi_L^*) \right\}$$

where the second term is necessary to ensure the action is real. This is known as a Majorana mass term. Write this explicitly in terms of η_a. Show that the resulting term actually vanishes if we interpret the field η as a classical object (an ordinary function), but is non-zero if we interpret η_a as an operator with anticommutation relations $\{eta_a, \eta_b\} = 0$.