PROBLEM SET 7
SOLUTIONS

(#2) We have:

$$A_i = \tfrac{1}{2}(J^i + iK^i) \qquad B_i = \tfrac{1}{2}(J^i - iK^i)$$

$$\begin{aligned}
[A_i, A_j] &= \tfrac{1}{4}[J_i + iK_i, J_j + iK_j] \\
&= \tfrac{1}{4}\left\{ [J_i, J_j] + i[J_i, K_j] + i[K_i, J_j] - [K_i, K_j] \right\} \\
&= \tfrac{1}{4}\left\{ i\varepsilon^{ijk} J_k + i(i\varepsilon^{ijk} K_k) - i(i\varepsilon^{jik} K_k) + i\varepsilon^{ijk} J_k \right\} \\
&= \tfrac{i}{4}\left\{ 2\varepsilon^{ijk} J_k + 2i\varepsilon^{ijk} K_k \right\} \\
&= i\varepsilon^{ijk}\left(\tfrac{1}{2} J_k + \tfrac{i}{2} K_k\right) \\
&= i\varepsilon^{ijk} A_k
\end{aligned}$$

Similarly, we find

$$[A_i, B_j] = 0 \qquad\qquad [B_i, B_j] = i\varepsilon_{ijk} B_k$$

(#3) a) Since the two subsystems are independent, we have

$$[J_A^i, J_B^j] = 0$$

(mathematically the full Hilbert space is a tensor product, and J_A^i acts on one factor while J_B^i acts on the other factor)

For each subsystem, J^i's are the rotation operators, so must satisfy the same commutation relations as usual:

$$[J_A^i, J_A^j] = i\varepsilon^{ijk} J_A^k \qquad\qquad [J_B^i, J_B^j] = i\varepsilon^{ijk} J_B^k$$

b) A basis of states would be:

$$|\uparrow\uparrow\rangle,\ |\uparrow\downarrow\rangle,\ |\downarrow\uparrow\rangle,\ |\downarrow\downarrow\rangle$$

or $|m_1\, m_2\rangle$ where $m_1 = \pm\frac{1}{2}$ and $m_2 = \pm\frac{1}{2}$.

We know:

$$J_A^i\,|m_1\, m_2\rangle = \tfrac{1}{2}(\sigma^i)_{m_1 n_1}\,|n_1\, m_2\rangle$$
$$J_B^i\,|m_1\, m_2\rangle = \tfrac{1}{2}(\sigma^i)_{m_2 n_2}\,|m_1\, n_2\rangle$$

So the matrix elements of J_A^i in this basis are:

$$\langle m_1\, m_2|\, J_A^i\, |n_1\, n_2\rangle = \tfrac{1}{2}\sigma^i_{m_1 n_1}\,\delta_{m_2 n_2}$$
$$\langle m_1\, m_2|\, J_B^i\, |n_1\, n_2\rangle = \tfrac{1}{2}\delta_{m_1 n_1}\,\sigma^i_{m_2 n_2}$$

Explicitly:

$$J_A^z = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} \qquad J_B^z = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

$$J_A^x = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} \qquad J_B^x = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

$$J_A^y = \begin{pmatrix} 0 & 0 & & -\frac{i}{2} \\ 0 & 0 & -\frac{i}{2} & 0 \\ 0 & +\frac{i}{2} & 0 & 0 \\ \frac{i}{2} & 0 & 0 & 0 \end{pmatrix} \qquad J_B^y = \begin{pmatrix} 0 & -\frac{i}{2} & 0 & 0 \\ \frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{2} \\ 0 & 0 & \frac{i}{2} & 0 \end{pmatrix}$$

We could have also derived these directly from how the J operators act on the states: eg.

$$J_A^z|\uparrow\uparrow\rangle = \tfrac{1}{2}|\uparrow\uparrow\rangle,\quad J_A^z|\uparrow\downarrow\rangle = \tfrac{1}{2}|\uparrow\downarrow\rangle,\quad J_A^z|\downarrow\uparrow\rangle = -\tfrac{1}{2}|\downarrow\uparrow\rangle,$$
$$J_A^z|\downarrow\downarrow\rangle = -\tfrac{1}{2}|\downarrow\downarrow\rangle$$

c) Since J_A^i and J_B^i have the same commutation relations as A^i and B^i, the matrices we have found give one specific representation of the Lorentz group. To find $\mathcal{J}^i$ and $\mathcal{K}^i$ for this representation, we just need

$$\mathcal{J}^i = A^i + B^i$$
$$\mathcal{K}^i = (A^i - B^i)\cdot\frac{1}{i}$$

We get:

$$\mathcal{J}^x = J_A^x + J_B^x = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \qquad \mathcal{K}^x = -i(J_A^x - J_B^x) = \begin{pmatrix} 0 & \frac{i}{2} & -\frac{i}{2} & 0 \\ \frac{i}{2} & 0 & 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 & 0 & \frac{i}{2} \\ 0 & -\frac{i}{2} & \frac{i}{2} & 0 \end{pmatrix}$$

$$\mathcal{J}^y = J_A^y + J_B^y = \begin{pmatrix} 0 & -\frac{i}{2} & 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 & -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} & 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 & \frac{i}{2} & 0 \end{pmatrix} \qquad \mathcal{K}^y = -i(J_A^y - J_B^y) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{pmatrix}$$

$$\mathcal{J}^z = J_A^z + J_B^z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \mathcal{K}^z = -i(J_A^z - J_B^z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

④ We need

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}\mathbb{1} \qquad (*)$$

a) From $(*)$ with $\mu=\nu=3$: $\gamma^3\gamma^3 = \eta^{33}\mathbb{1} = -\mathbb{1}$.

b) From $(*)$ with $\mu=1,\ \nu=2$: $\gamma^1\gamma^2+\gamma^2\gamma^1 = \eta^{12}\mathbb{1} = 0$.

c) From $(*)$: $\gamma^0\gamma^2 + \gamma^2\gamma^0 = 0$

$$\begin{aligned}
\Rightarrow \gamma^0\gamma^2\gamma^0 &= -\gamma^2\gamma^0\gamma^0 \\
&= -\gamma^2 \cdot \eta^{00} \cdot \mathbb{1} \\
&= -\gamma^2
\end{aligned}$$

d)
$$\begin{aligned}
\gamma^\mu\gamma^\nu\gamma_\mu &= \eta_{\mu\alpha}\gamma^\mu\gamma^\nu\gamma^\alpha \\
&= \eta_{\mu\alpha}\gamma^\mu(-\gamma^\alpha\gamma^\nu + 2\eta^{\nu\alpha}\mathbb{1}) \\
&= -\eta_{\mu\alpha}\cdot\tfrac{1}{2}(\gamma^\mu\gamma^\alpha+\gamma^\alpha\gamma^\mu)\cdot\gamma^\nu + 2\gamma^\mu\eta_{\mu\alpha}\cdot\eta^{\nu\alpha}\cdot\mathbb{1} \\
&= -\eta_{\mu\alpha}(\eta^{\mu\alpha}\cdot\mathbb{1})\cdot\gamma^\nu + 2\gamma^\nu \\
&= -4\gamma^\nu + 2\gamma^\nu \\
&= -2\gamma^\nu
\end{aligned}$$

For $v = \frac{3}{5}c$, we have $\beta = \frac{3}{5}$, $\gamma = \frac{5}{4}$ and

$$\Lambda = \begin{pmatrix} \frac{5}{4} & 0 & 0 & \frac{3}{4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{3}{4} & 0 & 0 & \frac{5}{4} \end{pmatrix}$$

$$\Lambda^{-1} x = \begin{pmatrix} \frac{5}{4} & 0 & 0 & -\frac{3}{4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{3}{4} & 0 & 0 & \frac{5}{4} \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{4}t - \frac{3}{4}z \\ x \\ y \\ -\frac{3}{4}t + \frac{5}{4}z \end{pmatrix}$$

The field transforms as:

$$\tilde{\psi}(x) = e^{ia \cdot K_z} \cdot \psi(\Lambda^{-1} x)$$

Now, $\psi(t, x, y, z) = (At, 0, 0, 0)$ so $\psi\left(\frac{5}{4}t - \frac{3}{4}z, x, y, -\frac{3}{4}t + \frac{5}{4}z\right)$

$$= \begin{pmatrix} A \cdot \frac{5}{4}t - A \cdot \frac{3}{4}z \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We have: $K_z = -\frac{i}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix}$ so $e^{iaK_z} = e^{\frac{a}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix}} = \begin{pmatrix} e^{\frac{a}{2}} & & & \\ & e^{-\frac{a}{2}} & & \\ & & e^{-\frac{a}{2}} & \\ & & & e^{\frac{a}{2}} \end{pmatrix}$

Finally: $\tanh(a) = \beta = \frac{3}{5}$ so $a = \ln(2)$ and $e^{\frac{a}{4}} = 2^{\frac{1}{4}}$.

Thus:

$$\tilde{\psi}(x) = \begin{pmatrix} 2^{\frac{1}{2}} & & & \\ & 2^{-\frac{1}{2}} & & \\ & & 2^{-\frac{1}{2}} & \\ & & & 2^{\frac{1}{2}} \end{pmatrix} \cdot \begin{pmatrix} A\left(\frac{5}{4}t - \frac{3}{4}z\right) \\ 0 \\ 0 \\ 0 \end{pmatrix} = 2^{\frac{1}{2}} A \begin{pmatrix} \frac{5}{4}t - \frac{3}{4}z \\ 0 \\ 0 \\ 0 \end{pmatrix}$$