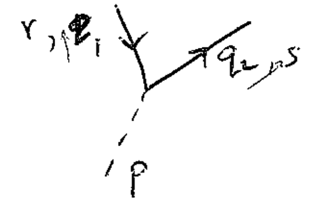
PROBLEM SET 11 SOLUTIONS

a) For the decay $\phi \to a^+ a^-$, the amplitude is

$i\mathcal{M} =$ 

from $-g_a \phi \bar{\psi} \psi$ interaction

$$= \bar{u}^r(q_2) \cdot v^s(q_1) \cdot (i g_a)$$

In the decay rate, we want $|\mathcal{M}|^2$, summed over the possible final particle spins. This gives

$$|\mathcal{M}|^2 = g_a^2 \sum_{r,s} \bar{u}^r(q_2) v^s(q_1) \bar{v}^s(q_1) u^r(q_2)$$

$$= g_a^2 \, \mathrm{Tr}\left((\not{q}_1 - m)(\not{q}_2 + m)\right)$$

$$= g_a^2 \left\{ q_{1\mu} q_{2\nu} \mathrm{Tr}(\gamma^\mu \gamma^\nu) - m^2 \mathrm{Tr}(\mathbb{1}) \right\}$$

$$= 4 g_a^2 (q_1 \cdot q_2 - m^2)$$

To find the decay rate, we work in the c.o.m. frame, where $p = (M_h, 0, 0, 0)$ $\quad q_1 = \left(\frac{M_h}{2}, \vec{q}\right)$ $\quad q_2 = \left(\frac{M_h}{2}, -\vec{q}\right)$

Then:

$$d\Gamma = |\mathcal{M}|^2 \frac{1}{2M_h} \cdot \frac{1}{2E_{q_1}} \frac{1}{2E_{q_2}} (2\pi)^4 \delta^4(q_1 + q_2 - p) \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_2}{(2\pi)^3}$$

We should integrate this over $\vec{q}_2$ and $|q_1|$ since these are fixed by the momentum conservation delta function.

Thus:

$$d\Gamma = 4 g_a^2 \Big(\underbrace{\frac{M^2}{4} + \vec{q}^{\,2} - m^2}_{q_1 \cdot q_2 - m^2}\Big) \cdot \frac{1}{2M} \cdot \frac{1}{M} \cdot \frac{1}{M} \cdot d\Omega$$

$$\int_0^\infty \frac{q_1^2\, dq_1}{(2\pi)^3} \int \frac{d^3\vec{q}_2}{(2\pi)^3} \cdot (2\pi)^3 \delta(\vec{q}_2 + \vec{q}_1) \cdot (2\pi)\, \delta\left(\sqrt{\vec{q}_1^{\,2} + m^2} + \sqrt{\vec{q}_2^{\,2} + m^2} - M\right)$$

$$= \frac{2 g_a^2}{M^3}\, d\Omega \left(\frac{M^2}{4} + \vec{q}^{\,2} - m^2\right) \int_0^\infty \frac{q_1^2\, dq_1}{(2\pi)^2} \cdot \delta\left(2\sqrt{q_1^2 + m^2} - M\right)$$

Now, $$\delta\left(2\sqrt{q_1^2 + m^2} - M\right) = \frac{\delta\left(q_1 - \sqrt{\left(\frac{M}{2}\right)^2 - m^2}\right)}{2q_1 / \sqrt{q_1^2 + m^2}}$$

So $$d\Gamma = \frac{g^2}{4\pi^2 M^2} \left(\left(\frac{M}{2}\right)^2 - m^2\right)^{\frac{3}{2}} d\Omega$$

To get the full decay rate, we integrate over the whole sphere (the particle & antiparticle are distinguishable) so

$$\boxed{\Gamma = \frac{g^2}{\pi M^2} \left(\left(\frac{M}{2}\right)^2 - m^2\right)^{\frac{3}{2}}}$$

b)

The ratio of decay rates for the electron and bottom is

$$\frac{\Gamma_e}{\Gamma_b} = \frac{g_e^2}{g_b^2} \frac{\left(\left(\frac{M}{2}\right)^2 - m_e^2\right)^{\frac{3}{2}}}{\left(\left(\frac{M}{2}\right)^2 - m_b^2\right)^{\frac{3}{2}}}$$

We have $g_e = \frac{m_e}{V}$ and $g_b = \frac{m_b}{V}$

so $\left(\frac{g_e}{g_b}\right) = \left(\frac{m_e}{m_b}\right) \approx \frac{0.5\,\text{MeV}}{4.5\,\text{GeV}}$

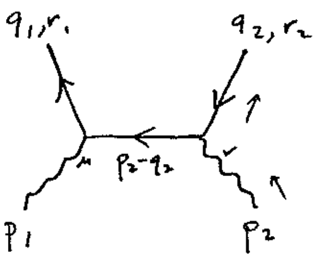
$$\approx \left(\frac{0.5}{4500}\right)^2 \cdot 1 \approx 10^{-8}$$

We have $\Gamma_b = \left(\frac{m_b}{V}\right)^2 \cdot \frac{1}{\pi M_h^2} \left(\left(\frac{M_h}{2}\right)^2 - m_b^2\right)^{\frac{3}{2}} = 1.9 \times 10^6\,\text{eV} = 2.9 \times 10^{21}\,\text{s}^{-1}.$

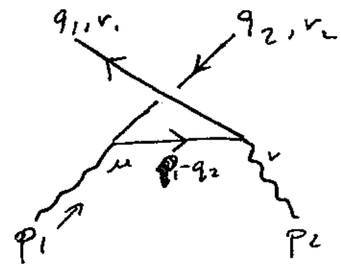
The lifetime is $1/\Gamma = \frac{1}{\Gamma_b + \Gamma_e} \approx \frac{1}{\Gamma_b}$ $\boxed{\approx 3.4 \times 10^{-22}\,\text{s}}$

1) We have 2 diagrams:

I



II



The amplitudes are:

$$i\mathcal{M}_I = (-ie)^2 \varepsilon_\mu(p_1)\varepsilon_\nu(p_2)\,\bar{u}^{r_1}(q_1)\gamma^\mu \Delta_F(p_2-q_2)\gamma^\nu v^{r_2}(q_2)$$

$$i\mathcal{M}_{II} = (-ie)^2 \varepsilon_\mu(p_1)\varepsilon_\nu(p_2)\,\bar{u}^{r_1}(q_1)\gamma^\nu \Delta_F(q_1-q_2)\gamma^\mu v^{r_2}(q_2)$$

So
$$i\mathcal{M} = -ie^2 \varepsilon_\mu(p_1)\varepsilon_\nu(p_2)\,\bar{u}^{r_1}(q_1)\left[\underbrace{\frac{\gamma^\mu(\not{p}_2-\not{q}_2+m)\gamma^\nu}{(p_2-q_2)^2-m^2}}_{A^{\mu\nu}} + \underbrace{\frac{\gamma^\nu(\not{p}_1-\not{q}_2+m)\gamma^\mu}{(p_1-q_2)^2-m^2}}_{B^{\nu\mu}}\right] v^{r_2}(q_2)$$

The squared amplitude is (after summing/averaging over final/initial particle spins + polarizations

$$\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{1}{4}e^4\, \mathrm{tr}\Big(\big(A^{\mu\nu}+B^{\nu\mu}\big)(\not{q}_2-m)\big(\tilde{A}^{\tilde{\nu}\tilde{\mu}}+\tilde{B}^{\tilde{\mu}\tilde{\nu}}\big)(\not{q}_1+m)\Big)\eta_{\mu\tilde{\mu}}\eta_{\nu\tilde{\nu}}$$

This is the same as for Compton scattering, with the replacements

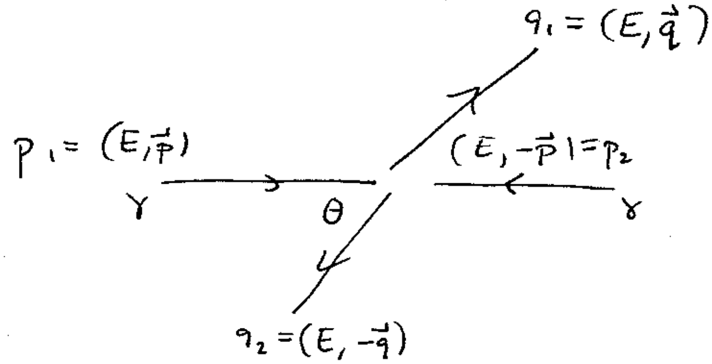
$$p' \to q_1\,,\quad k \to p_2\,,\quad p \to -q_2\,,\quad k' \to -p_1$$

after we remove an overall $-$ sign. Thus we find

$$\frac{1}{4}\sum|\mathcal{M}|^2 = 2e^4\left[+\frac{p_1\cdot q_2}{p_2\cdot q_2} + \frac{p_2\cdot q_2}{p_1\cdot q_2} + 2m^2\left(\frac{1}{p_1\cdot q_2}+\frac{1}{p_2\cdot q_2}\right) - m^4\left(\frac{1}{p_1\cdot q_2}+\frac{1}{p_2\cdot q_2}\right)^2\right]$$

The kinematics (ie the rest of the cross section formula not determined by the QFT interactions) is the same as for the $\phi\phi\to e^+e^-$ from last assignment, or for massless $(e^+e^-)\to\mu^+\mu^-$. Thus, we find

(see next)



$$\frac{d\sigma}{d\Omega} = \frac{1}{8E^2} \cdot \frac{|\vec{q}|}{16\pi^2(2E)} \left(\frac{1}{4}\sum|M|^2\right)$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{8E^2}\frac{|\vec{q}|}{32\pi^2 E} \cdot 2e^4 \left[\frac{E^2+\vec{p}\cdot\vec{q}}{E^2-\vec{p}\cdot\vec{q}} + \frac{E^2-\vec{p}\cdot\vec{q}}{E^2+\vec{p}\cdot\vec{q}} + 2m^2\left(\frac{1}{E^2+\vec{p}\cdot\vec{q}} + \frac{1}{E^2-\vec{p}\cdot\vec{q}}\right) - m^4\left(\frac{1}{E^2+\vec{p}\cdot\vec{q}} + \frac{1}{E^2-\vec{p}\cdot\vec{q}}\right)^2\right]$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2} \cdot \beta \cdot \left[\frac{1+\beta^2\cos^2\theta}{1-\beta^2\cos^2\theta} + 2\left(\frac{m}{E}\right)^2 \frac{1}{1-\beta^2\cos^2\theta} - 2\left(\frac{m}{E}\right)^4 \frac{1}{(1-\beta^2\cos^2\theta)^2}\right]$$

Here, $\beta = \frac{|\vec{q}|}{E} = \sqrt{1-\left(\frac{m}{E}\right)^2}$. The total cross section is

$$\sigma = 2\pi\int_0^\pi \sin\theta \, d\theta \, \frac{d\sigma}{d\Omega}$$

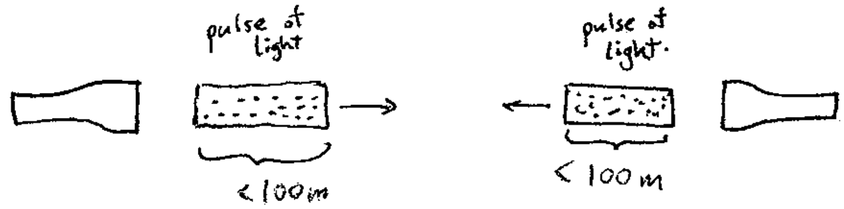
Using $\beta = \sqrt{1-\left(\frac{0.511\,\text{MeV}}{1\,\text{MeV}}\right)^2}$ and $\left(\frac{m}{E}\right) = (0.511)$, $\alpha = \frac{1}{137}$

and $\frac{1}{E^2} = \frac{1}{(1\,\text{MeV})^2} = 3.894 \times 10^{-22}\,\text{cm}^2$, we find

$$\cancel{\sigma = 1.156 \times 10^{-23}\,\text{cm}^2}$$

$$\boxed{\sigma = 1.360 \times 10^{-25}\,\text{cm}^2}$$

Now, the minimum energy will be if the power is large enough so that the lengths of the required pulses of light are less than the distance between the sources, so all photons from one source have an opportunity to interact with all photons from the other source.



Each photon from the one source will react with a fraction $\frac{\sigma}{A}$ of the photons from the other source to produce an e^+e^- pair. Thus, we want

$$N^2 \cdot \frac{\sigma}{A} = 1{,}000{,}000$$

where N is the number of photons in each pulse. The total energy is $2N \cdot 1\,\text{MeV}$, so

$$E = 2\,\text{MeV} \times (1.602 \times 10^{-13}\,\text{J/MeV}) \times 1000 \times \sqrt{\frac{A}{\sigma}}$$

$$\Rightarrow \boxed{E = 86.89\ \text{Joules}}$$

Note: to produce a steady 1,000,000 ~~[illegible]~~ positrons/sec, we would need a total power such that

$$\frac{\sigma}{A} \cdot \left(\frac{2L}{c}\left(\frac{P/2}{E}\right)\right)\left(\frac{P/2}{E}\right) = 1{,}000{,}000/\text{s}$$

$$\Rightarrow \boxed{P = 1.06\ \text{MW}}$$