PROBLEM SET 4 SOLUTIONS

① For a single harmonic oscillator,

$$x(t) = \frac{1}{\sqrt{2\omega}}\left(a e^{-i\omega t} + a^{\dagger} e^{i\omega t}\right)$$

We have

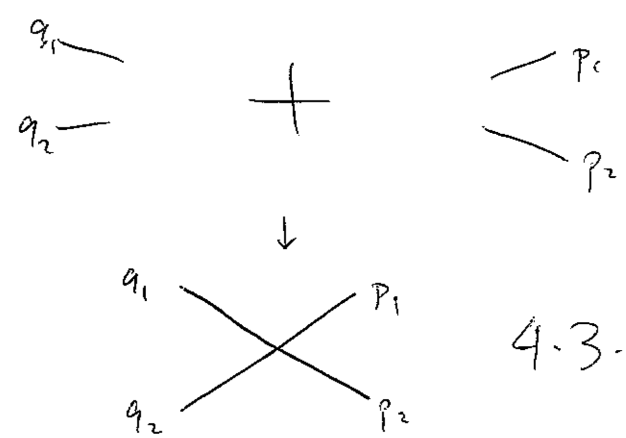
$$D_F(t_1 - t_2) = \langle 0 | T\{x(t_1)\, x(t_2)\} | 0 \rangle$$

$$= \theta(t_1 - t_2)\langle 0 | x(t_1) x(t_2) | 0\rangle + \theta(t_2 - t_1)\langle 0 | x(t_2)\, x(t_1) | 0 \rangle$$

$$= \frac{1}{2\omega}\left\{ \theta(t_1 - t_2)\, e^{i\omega(t_2 - t_1)} + \theta(t_2 - t_1)\, e^{i\omega(t_1 - t_2)} \right\}$$

③ $$-i\lambda \int_{-\infty}^{\infty} dt \int d^3x \ \langle 0 | a_{q_1} a_{q_2}\, \phi(x,t)\,\phi(x,t)\,\phi(x,t)\,\phi(x,t)\, a^{\dagger}_{p_1} a^{\dagger}_{p_2} | 0 \rangle$$

to calculate this, we have:



$4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways

So we get $$-i\lambda \cdot 24 \cdot \int_{-\infty}^{\infty} dt \int d^3x \ \langle a_{q_1} \phi \rangle \langle a_{q_2} \phi \rangle \langle \phi\, a^{\dagger}_{p_1} \rangle \langle \phi\, a^{\dagger}_{p_2} \rangle$$

$$= -24 i \lambda \int d^4x \ \frac{1}{\sqrt{2\omega_{q_1}}} e^{iq_1 \cdot x} \frac{1}{\sqrt{2\omega_{q_2}}} e^{iq_2 \cdot x} \frac{1}{\sqrt{2\omega_{p_1}}} e^{-ip_1 \cdot x} \frac{1}{\sqrt{2\omega_{p_2}}} e^{-ip_2 \cdot x}$$

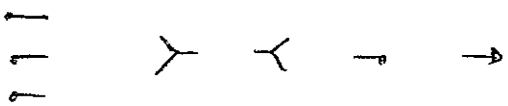
$$= -6 i \lambda \cdot \frac{(2\pi)^4}{\sqrt{\omega_{q_1}\omega_{q_2}\omega_{p_1}\omega_{p_2}}} \cdot \delta(q_1 + q_2 - p_1 - p_2)$$

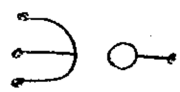
2. Transition amplitude =

$$\langle 0|\, aaa \frac{1}{\sqrt{3!}} \left\{ -\frac{1}{2}\int_0^T dt_1 \int_0^T dt_2\, T\{x^3(t_1)\, x^3(t_2)\}\left(\frac{\lambda}{3!}\right)^2\right\} a^\dagger |0\rangle$$

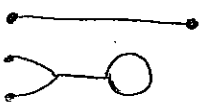
$$= -\frac{1}{2}\frac{1}{\sqrt{3!}}\left(\frac{\lambda}{3!}\right)^2 \int_0^T dt_1 \int_0^T dt_2 \;\langle 0|\, aaa\, T\{x(t_1)x(t_1)x(t_1)x(t_2)x(t_2)x(t_2)\}\, a^\dagger|0\rangle$$

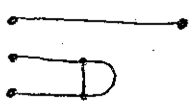
The correlator has the following contributions:

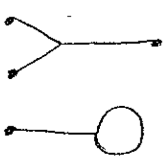


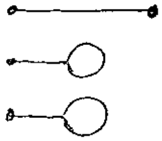
①  $6\times 2\times 3\cdot\left(\frac{1}{\sqrt{2\omega}}e^{i\omega t_1}\right)^3\left(\frac{1}{\sqrt{2\omega}}e^{-i\omega t_2}\right)D_F(0)$

②  $3\cdot 6\cdot 3\cdot 2\cdot 2\ \left(\frac{1}{\sqrt{2\omega}}e^{it_1\omega}\right)^2\left(\frac{1}{\sqrt{2\omega}}e^{i\omega t_2}\right)\left(\frac{1}{\sqrt{2\omega}}e^{-i\omega t_2}\right)D_F(t_1-t_2)$

③  $3\cdot 6\cdot 2\cdot 3\cdot(1)\cdot\left(\frac{1}{\sqrt{2\omega}}e^{it_1\omega}\right)^2 D_F(0)\,D_F(t_1-t_2)$

④  $3\cdot 6\cdot 3\cdot 2\cdot(1)\cdot\left(\frac{1}{\sqrt{2\omega}}e^{it_1\omega}\right)\left(\frac{1}{\sqrt{2\omega}}e^{it_2\omega}\right)D_F^2(t_1-t_2)$

⑤  $3\cdot 6\cdot 3\cdot 2\cdot\left(\frac{1}{\sqrt{2\omega}}e^{it_1\omega}\right)^2\left(\frac{1}{\sqrt{2\omega}}e^{it_2\omega}\right)\left(\frac{1}{\sqrt{2\omega}}e^{-it_1\omega}\right)D_F(0)$

⑥  $3\cdot 6\cdot 3\ (1)\cdot\left(\frac{1}{\sqrt{2\omega}}e^{it_1\omega}\right)\left(\frac{1}{\sqrt{2\omega}}e^{it_2\omega}\right)D_F^2(0)$

Combining these, and using $D_F(0) = \frac{1}{2\omega}$, we get

$$\text{Amplitude} = -\frac{\lambda^2}{2\sqrt{6}} \int_0^T dt_1 \int_0^T dt_2 \left\{ \frac{1}{8\omega^3} \left(e^{3i\omega t_1 - i\omega t_2} + \frac{9}{2} e^{i\omega(t_1+t_2)} \right) \right.$$

$$+ \frac{9}{4\omega^2} \, e^{2i\omega t_1} D_F(t_1 - t_2)$$

$$\left. + \frac{3}{2\omega} \, e^{i\omega(t_1+t_2)} D_F^2(t_1 - t_2) \right\}$$

$$= -\frac{\lambda^2}{2\sqrt{6}} \frac{1}{8\omega^3} \int_0^T dt_1 \, e^{3i\omega t_1} \int_0^T dt_2 \, e^{-i\omega t_2}$$

$$- \frac{\lambda^2}{2\sqrt{6}} \frac{1}{8\omega^3} \frac{9}{2} \int_0^T dt_1 \, e^{i\omega t_1} \int_0^T dt_2 \, e^{i\omega t_2}$$

$$- \frac{\lambda^2}{2\sqrt{6}} \frac{9}{8\omega^3} \left[\int_0^T dt_1 \int_0^{t_1} dt_2 \left(e^{i\omega t_1} \cdot e^{i\omega t_2} \right) + \int_0^T dt_1 \int_{t_1}^T dt_2 \, e^{3i\omega t_1} e^{-i\omega t_2} \right]$$

$$- \frac{\lambda^2}{2\sqrt{6}} \frac{3}{8\omega^3} \left[\int_0^T dt_1 \int_0^{t_1} dt_2 \, e^{3i\omega t_2} e^{-i\omega t_1} + \int_0^T dt_1 \int_{t_1}^T dt_2 \, e^{3i\omega t_1} e^{-i\omega t_2} \right]$$

$$= -\frac{\lambda^2}{16\sqrt{6}\,\omega^3} \left[\frac{1}{3i\omega} \left(e^{3i\omega T} - 1 \right) \frac{1}{-i\omega} \left(e^{-i\omega T} - 1 \right) \right.$$

$$+ \frac{9}{2} \frac{1}{i\omega} \left(e^{i\omega T} - 1 \right) \frac{1}{i\omega} \left(e^{i\omega T} - 1 \right)$$

$$+ 9 \cdot \frac{1}{i\omega} \left[\frac{1}{2i\omega} \left(e^{2i\omega T} - 1 \right) - \frac{1}{i\omega} \left(e^{i\omega T} - 1 \right) \right]$$

$$+ 9 \cdot \frac{1}{-i\omega} \left[\frac{1}{3i\omega} \left(e^{2i\omega T} - e^{-i\omega T} \right) - \frac{1}{2i\omega} \left(e^{2i\omega T} - 1 \right) \right]$$

$$+ 3 \, \frac{1}{3i\omega} \left[\frac{1}{2i\omega} \left(e^{2i\omega T} - 1 \right) - \frac{1}{-i\omega} \left(e^{-i\omega T} - 1 \right) \right]$$

$$\left. + 3 \, \frac{1}{-i\omega} \left[\frac{1}{3i\omega} \left(e^{2i\omega T} - e^{-i\omega T} \right) - \frac{1}{2i\omega} \left(e^{2i\omega T} - 1 \right) \right] \right]$$

$$= -\frac{\lambda^2}{16\sqrt{6}\,\omega^5} \left[-\frac{16}{3} e^{-i\omega T} - \frac{7}{6} + 18 \, e^{i\omega T} - \frac{67}{6} e^{2i\omega T} - \frac{1}{3} e^{3i\omega T} \right]$$