PHYSICS 402 WORKSHEET

We've seen that there is a general relation between symmetries and conserved quantities in quantum mechanics. In this worksheet, we'll use this to understand proeerties of the momentum operator for a particle in one dimension

① Let $\hat{\tau}(a)$ be the unitary operator that tells us how the quantum state transforms when we translate the system to the right by a. For small a, let

$$\hat{\tau} = \mathbb{1} - i\frac{a}{\hbar}\hat{P} + \dots \qquad (*)$$

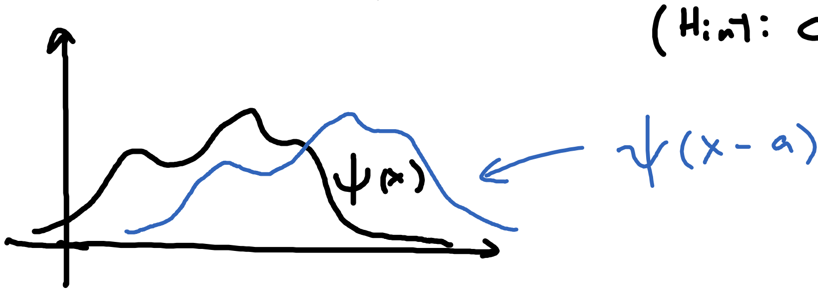
We know that $\hat{P}$ corresponds to a conserved quantity for systems with translation symmetry. We call this the MOMENTUM operator.

a) If the position wavefunction of a state $|\Psi\rangle$ is

$$\langle x|\Psi\rangle = \psi(x)$$

what is the position wavefunction for $\hat{\tau}(a)|\Psi\rangle$?

(Hint: can you draw it?)



b) Using this and (*), what is the wavefunction for the state $\hat{P}|\Psi\rangle$? Have $\psi(x-a) = \langle x|\hat{\tau}(a)|\psi\rangle$

Taylor expand around a=0: $\Rightarrow \psi(x) - a\frac{d\psi}{dx} + \dots = \langle x|(\mathbb{1} - \frac{i}{\hbar}\hat{P}a + \dots)|\psi\rangle$

$$= \psi(x) - \frac{i}{\hbar}a\langle x|\hat{P}|\psi\rangle + \dots$$

$$\Rightarrow \langle x|\hat{P}|\psi\rangle = \frac{\hbar}{i}\frac{d}{dx}\psi(x)$$

c) Using your result, complete the following sentence:
"Acting with $\hat{P}$ on a state corresponds to acting with $\hbar/i \ \frac{d}{dx}$ on the wavefunction.

d) What is the wavefunction for $\hat{X}|\Psi\rangle$, where $\hat{X}$ is the position operator whose eigenstates are $|x\rangle$?

Wavefn. is $\langle x|\hat{X}|\Psi\rangle = x\langle x|\Psi\rangle = x\,\psi(x)$. Here we have used that since $\hat{X}$ is Hermitian, we can act "to the left" on $\langle x|$ to give $\langle x|\hat{X} = x\langle x|$. In more detail, this can be derived as:

$$\langle x|\hat{X}|\Psi\rangle = \langle\Psi|\hat{X}^\dagger|x\rangle^* = \langle\Psi|\hat{X}|x\rangle^* = \langle\Psi|x|x\rangle^* = x\langle x|\Psi\rangle$$

e) Suppose we have a particle in a potential $V(x)$ so that it's classical energy is

$$E = \frac{p^2}{2m} + V(x)$$

In the quantum description of this system, we might expect that the energy operator should be:

$$\hat{H} = \frac{1}{2m}\hat{P}^2 + V(\hat{X})$$

In this case, the Schrödinger equation reads

$$i\hbar \frac{d}{dt}|\Psi\rangle = \left(\frac{1}{2m}\hat{P}^2 + V(\hat{X})\right)|\Psi\rangle$$

By taking the inner product of this with $\langle x|$ and using your results above, derive the equation for how the particle's wavefunction evolves with time.

We get $i\hbar \frac{d}{dt}\langle x|\Psi\rangle = \frac{1}{2m}\langle x|\hat{P}^2|\Psi\rangle + \langle x|V(\hat{x})|\Psi\rangle$

Using $\langle x|\hat{P}^2|\Psi\rangle = \frac{\hbar}{i}\frac{d}{dx}\langle x|\hat{P}|\Psi\rangle = \left(\frac{\hbar}{i}\frac{d}{dx}\right)\left(\frac{\hbar}{i}\frac{d}{dx}\right)\langle x|\Psi\rangle = -\hbar^2\frac{d^2\psi}{dx^2}$

and $\langle x|V(\hat{x})|\Psi\rangle = V(x)\langle x|\Psi\rangle = V(x)\psi(x)$

we get: $\boxed{i\hbar\frac{d}{dt}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{d^2\psi(x,t)}{dx^2} + V(x)\psi(x,t)}$

f) What is the wavefunction for $\hat{X}\hat{P}|\Psi\rangle$?

It is $\langle x|\hat{X}\hat{P}|\Psi\rangle = x\langle x|\hat{P}|\Psi\rangle = x\cdot\frac{\hbar}{i}\frac{d\psi}{dx}$

g) Using d) and c), what is the wavefunction for $\hat{P}\hat{X}|\Psi\rangle$?

It is $\frac{\hbar}{i}\frac{d}{dx}\left(x\psi(x)\right) = \frac{\hbar}{i}\psi(x) + \frac{\hbar}{i}x\frac{d\psi}{dx}$

h) Using f) and g), what is the wavefunction for $[\hat{X},\hat{P}]|\psi\rangle$? What can you conclude about the operator $[\hat{X},\hat{P}]$

We have: $\langle x|[\hat{X},\hat{P}]|\psi\rangle = \langle x|\hat{X}\hat{P}|\psi\rangle - \langle x|\hat{P}\hat{X}|\psi\rangle$

$$= x\frac{\hbar}{i}\frac{d\psi}{dx} - \frac{\hbar}{i}x\frac{d\psi}{dx} - \frac{\hbar}{i}\psi(x)$$

$$= i\hbar\,\psi(x)$$

We conclude $[\hat{X},\hat{P}] = i\hbar$

i) What does the generalized uncertainty principle

$$\Delta A\,\Delta B \geq \left|\langle\psi|\tfrac{1}{2i}[\hat{A},\hat{B}]|\psi\rangle\right|$$ tell us in this case?

We get $\Delta X\,\Delta P \geq \left|\langle\psi|\frac{1}{2i}\cdot i\hbar|\psi\rangle\right| = \frac{\hbar}{2}$

j) Can you show that for any state,

$$m \frac{d}{dt} \langle \hat{X} \rangle = \langle \hat{P} \rangle \ ?$$

We have $m \frac{d}{dt} \langle \psi | \hat{X} | \psi \rangle$

$$= m \left(\frac{d \langle \psi |}{dt} \right) \hat{X} | \psi \rangle + m \langle \psi | \hat{X} \left(\frac{d}{dt} | \psi \rangle \right)$$

Using $\frac{d|\psi\rangle}{dt} = \frac{-i\hat{H}}{\hbar} |\psi\rangle$, we get $\frac{d\langle\psi|}{dt} = \frac{i}{\hbar} \langle \psi | \hat{H}$

So $m \frac{d}{dt} \langle \psi | \hat{X} | \psi \rangle$

$$= \frac{mi}{\hbar} \langle \psi | \left(\hat{H}\hat{X} - \hat{X}\hat{H} \right) | \psi \rangle$$

$$= \frac{mi}{\hbar} \langle \psi | \left(\frac{\hat{P}^2}{2m} + V(\hat{X}) \right) \hat{X} - \hat{X} \left(\frac{\hat{P}^2}{2m} + V(\hat{X}) \right) | \psi \rangle$$

Now $[\hat{X}, V(\hat{X})] = 0$ and $\hat{P}^2 \hat{X} - \hat{X} \hat{P}^2$

$$= \hat{P}^2\hat{X} - \hat{P}\hat{X}\hat{P} + \hat{P}\hat{X}\hat{P} - \hat{X}\hat{P}^2$$

$$= -\hat{P}[\hat{X}, \hat{P}] - [\hat{X}, \hat{P}]\hat{P}$$

$$= -2i\hbar \hat{P}$$

Thus $m \frac{d}{dt} \langle \psi | \hat{X} | \psi \rangle = \langle \psi | \hat{P} | \psi \rangle$