PHYSICS 402 WORKSHEET

We've seen that there is a general relation between symmetries and conserved quantities in quantum mechanics. In this worksheet, we'll use this to understand proserties of the momentum operator for a particle in one dimension

① Let $\hat{\tau}(a)$ be the unitary operator that tells us how the quantum state transforms when we translate the system to the right by a. For small a, let

$$\hat{\tau} = \mathbb{1} - i\frac{a}{\hbar}\hat{P} + \ldots \qquad (*)$$

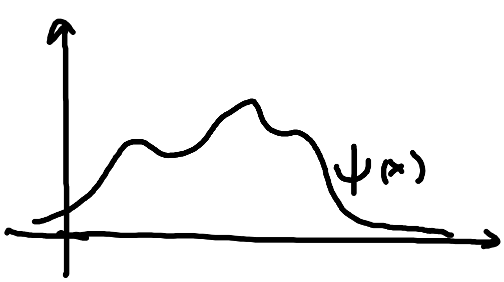
We know that $\hat{P}$ corresponds to a conserved quantity for systems with translation symmetry. We call this the MOMENTUM operator.

a) If the position wavefunction of a state $|\Psi\rangle$ is

$$\langle x|\Psi\rangle = \psi(x)$$

what is the position wavefunction for $\hat{\tau}(a)|\Psi\rangle$?

(Hint: can you draw it?)



b) Using this and (*), what is the wavefunction for the state $\hat{P}|\Psi\rangle$?

c) Using your result, complete the following sentence:
"Acting with $\hat{P}$ on a state corresponds to acting with ________________ on the wavefunction.

d) What is the wavefunction for $\hat{X}|\Psi\rangle$, where $\hat{X}$ is the position operator whose eigenstates are $|x\rangle$?

e) Suppose we have a particle in a potential $V(x)$ so that it's classical energy is

$$E = \frac{p^2}{2m} + V(x)$$

In the quantum description of this system, we might expect that the energy operator should be:

$$\hat{H} = \frac{1}{2m}\hat{P}^2 + V(\hat{X})$$

In this case, the Schrödinger equation reads

$$i\hbar \frac{d}{dt}|\Psi\rangle = \left(\frac{1}{2m}\hat{P}^2 + V(\hat{X})\right)|\Psi\rangle$$

By taking the inner product of this with $\langle x|$ and using your results above, derive the equation for how the particle's wavefunction evolves with time.

f) What is the wavefunction for $\hat{X}\hat{P}|\Psi\rangle$?

g) What is the wavefunction for $\hat{P}\hat{X}|\Psi\rangle$? Hint: apply the result c) to state $\hat{X}|\Psi\rangle$ (also use d)).

h) Using f) and g), what is the wavefunction for $[\hat{X},\hat{P}]|\psi\rangle$? What can you conclude about the operator $[\hat{X},\hat{P}]$

i) What does the generalized uncertainty principle $\Delta A \Delta B \geq |\langle\psi|\frac{1}{2i}[\hat{A},\hat{B}]|\psi\rangle|$ tell us in this case?

j) Can you show that for any state,

$$m \frac{d}{dt} \langle \hat{X} \rangle = \langle \hat{P} \rangle \; ?$$