Worksheet on Operators

In this worksheet, we'll think about an example of an operator acting on a 2D Hilbert space.

We'll use $|\uparrow\rangle$ and $|\downarrow\rangle$ to represent orthonormal basis elements for the space.

Since the action of an operator is linear, it is fully specified by saying what it does to the states in a basis. So we will define our operator $\mathcal{O}$ by saying that:

$$\mathcal{O}|\uparrow\rangle = 3|\uparrow\rangle + |\downarrow\rangle$$

$$\mathcal{O}|\downarrow\rangle = |\uparrow\rangle + 3|\downarrow\rangle$$

a) If $|\Psi\rangle = \frac{3}{5}|\uparrow\rangle + \frac{4}{5}i|\downarrow\rangle$, then $\mathcal{O}|\Psi\rangle = A|\uparrow\rangle + B|\downarrow\rangle$
What are A and B?

We have: $\mathcal{O}|\Psi\rangle = \mathcal{O}\left(\frac{3}{5}|\uparrow\rangle + \frac{4}{5}i|\downarrow\rangle\right)$

$$= \frac{3}{5}\mathcal{O}|\uparrow\rangle + \frac{4}{5}i\,\mathcal{O}|\downarrow\rangle$$

$$= \frac{3}{5}\left(3|\uparrow\rangle + |\downarrow\rangle\right) + \frac{4}{5}i\left(|\uparrow\rangle + 3|\downarrow\rangle\right)$$

$$= \underbrace{\left(\frac{9}{5} + \frac{4}{5}i\right)}_{A}|\uparrow\rangle + \underbrace{\left(\frac{3}{5} + \frac{12}{5}i\right)}_{B}|\downarrow\rangle$$

b) If we represent a state $|\Psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ by the column vector $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, then the action of $\mathcal{O}$ on a state is represented by

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \rightarrow \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

What are a, b, c, and d?

$$\mathcal{O}(\alpha|\uparrow\rangle + \beta|\downarrow\rangle)$$
$$= \alpha(3|\uparrow\rangle + 1|\downarrow\rangle) + \beta(|\uparrow\rangle + 3|\downarrow\rangle)$$
$$= (3\alpha+\beta)|\uparrow\rangle + (\alpha+3\beta)|\downarrow\rangle$$

So: $\alpha \to 3\alpha + \beta$
$\beta \to \alpha + 3\beta$

i.e. $\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \to \begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ so $\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}$

c) The operator $\mathcal{O}$ has two independent eigenvectors with corresponding eigenvalues λ_1, λ_2. What are λ_1 and λ_2?

characteristic polynomial $\lambda^2 - \mathrm{tr}\,\mathcal{O}\,\lambda + \det\mathcal{O} = 0$

$$\lambda^2 - 6\lambda + 8 = 0$$
$$(\lambda-2)(\lambda-4) = 0$$
$$\lambda_1, \lambda_2 = 2, 4$$

d) What are the eigenvectors corresponding to these eigenvalues? Write them as $A|\uparrow\rangle + B|\downarrow\rangle$.

Can use $\begin{pmatrix}b\\ \lambda - a\end{pmatrix}$ for eigenvector of $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ w. eigenvalue λ.

So: $\lambda = 2$: $\begin{pmatrix}1\\ -1\end{pmatrix}$ $\lambda = 4$: $\begin{pmatrix}1\\ 1\end{pmatrix}$

e) Can you describe geometrically the action of $\mathcal{O}$?