Worksheet on Operators

In this worksheet, we'll think about an example of an operator acting on a 2D Hilbert space.

We'll use $|\uparrow\rangle$ and $|\downarrow\rangle$ to represent orthonormal basis elements for the space.

Since the action of an operator is linear, it is fully specified by saying what it does to the states in a basis. So we will define our operator $\mathcal{O}$ by saying that:

$$\mathcal{O}|\uparrow\rangle = 3|\uparrow\rangle + |\downarrow\rangle$$

$$\mathcal{O}|\downarrow\rangle = |\uparrow\rangle + 3|\downarrow\rangle$$

a) If $|\Psi\rangle = \frac{3}{5}|\uparrow\rangle + \frac{4}{5}i|\downarrow\rangle$, then $\mathcal{O}|\Psi\rangle = A|\uparrow\rangle + B|\downarrow\rangle$
What are A and B?

b) If we represent a state $|\Psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ by the column vector $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, then the action of $\hat{O}$ on a state is represented by

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \rightarrow \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

What are $a, b, c,$ and d?

c) The operator $\hat{O}$ has two independent eigenvectors with corresponding eigenvalues λ_1, λ_2. What are λ_1 and λ_2?

d) What are the eigenvectors corresponding to these eigenvalues? Write them as $A|\uparrow\rangle + B|\downarrow\rangle$.

e) Can you describe geometrically the action of $\hat{O}$?