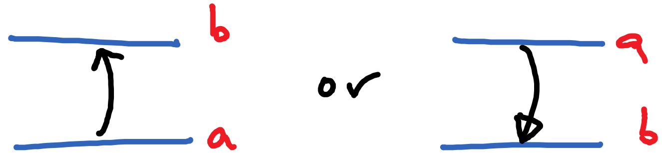
LAST TIME: Rate for ABSORPTION or SPONTANEOUS EMISSION



$$R_{a\to b} = \frac{\pi}{3\varepsilon_0 \hbar^2}\left|\vec{\wp}_{ba}\right|^2 \rho(\omega_0)$$

$\langle \psi_b | \sum q_i \vec{x}_i | \psi_a \rangle$

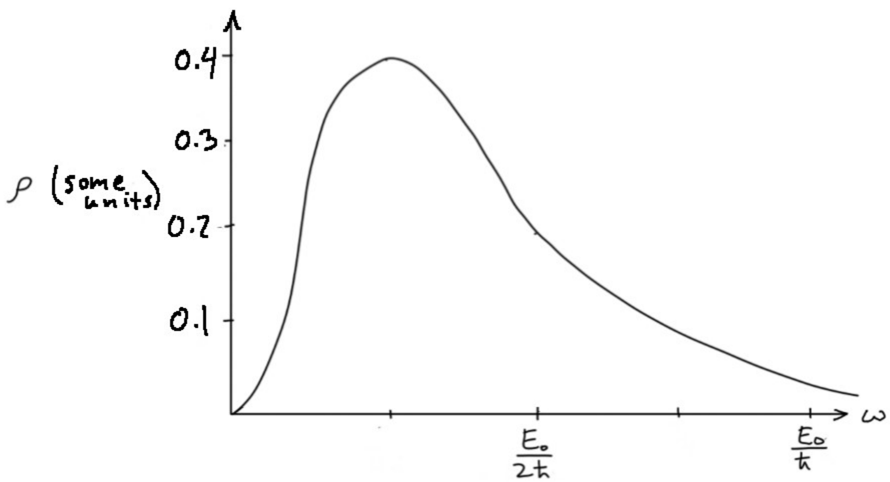
energy density of radiation (per unit frequency)

$\dfrac{|E_b - E_a|}{\hbar}$

A particular molecule has three low-energy states $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$ with energies $-E_0$, $-E_0/2$, and $-E_0/4$ respectively. The matrix elements $\langle\psi_a|\vec{\mathcal{P}}|\psi_b\rangle$ of the electric dipole moment operator for these three states are

$$\mathcal{P}^x_{ab} = P_0 \begin{pmatrix} 1 & 2+i & 0 \\ 2-i & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \mathcal{P}^y_{ab} = P_0 \begin{pmatrix} 0 & -i & 0 \\ i & 3 & 2 \\ 0 & 2 & 0 \end{pmatrix} \qquad \mathcal{P}^z_{ab} = P_0 \begin{pmatrix} 0 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A collection of 100,000 of these molecules are prepared in the $|\psi_2\rangle$ state. The molecules are in an environment where the background radiation density is as shown in the plot.



What is the expected amount of time before 10 molecules are in the state $|\psi_3\rangle$? You can ignore processes with multiple transitions (e.g. 2 -> 1 -> 3).

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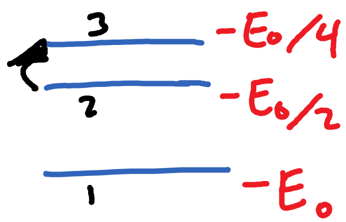

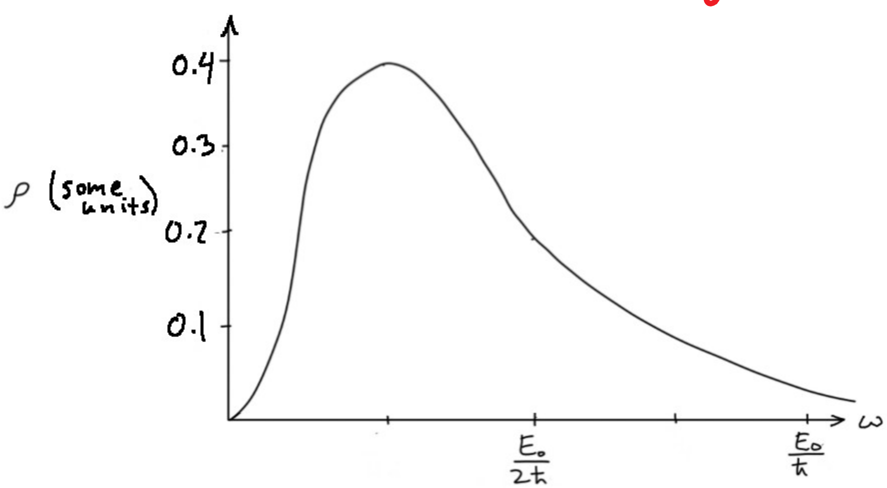

What is the expected amount of time before 10 molecules are in the state $|\psi_3\rangle$? You can ignore processes with multiple transitions (e.g. 2 -> 1 -> 3).

Q: Which process do we care about here?

1. Absorption $1 \to 2$
2. Absorption $2 \to 3$
3. Stimulated emission $3 \to 2$
4. Stimulated emission $2 \to 1$
5. More than one of these

$$R_{a\to b} = \frac{\pi}{3\varepsilon_0 \hbar^2} \left| \vec{\mathcal{P}}_{ab} \right|^2 \rho(\omega_0)$$

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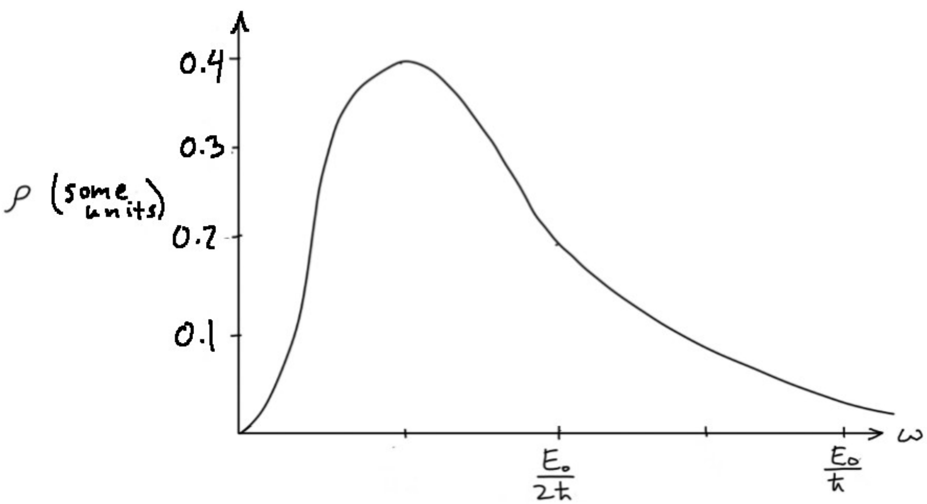

What is the expected amount of time before 10 molecules are in the state $|\psi_3\rangle$? You can ignore processes with multiple transitions (e.g. 2 -> 1 -> 3).

Q: What is $\rho(\omega_0)$ for $R_{2\to 3}$?

$$R_{a\to b} = \frac{\pi}{3\varepsilon_0 \hbar^2} \left| \vec{\mathcal{P}}_{ab} \right|^2 \rho(\omega_0)$$

$$\frac{|E_3 - E_2|}{\hbar} = \frac{E_0}{4\hbar}$$

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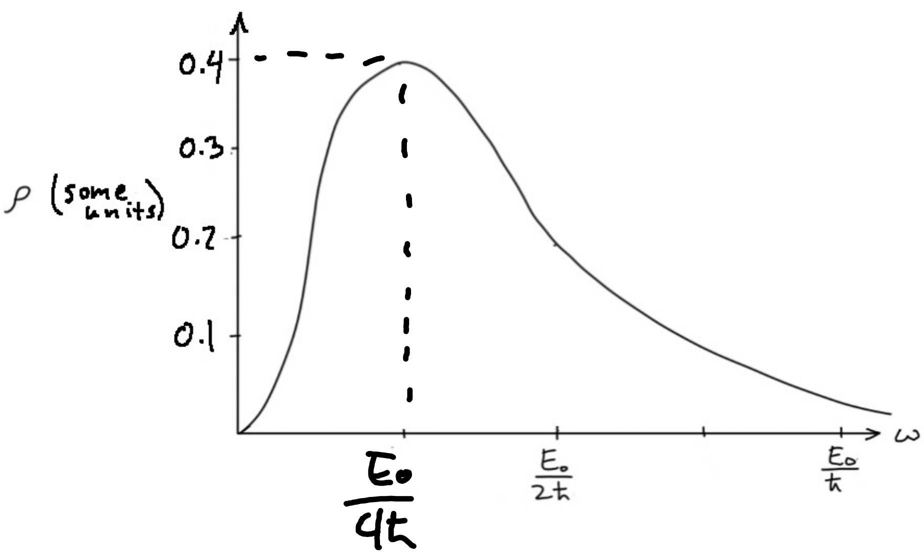

What is the expected amount of time before 10 molecules are in the state $|\psi_3\rangle$? You can ignore processes with multiple transitions (e.g. 2 -> 1 -> 3).

Q: What is $\rho(\omega_0)$ for $R_{2\to 3}$?

① 0.1 ② 0.2 ③ 0.3

④ 0.4

$$R_{a\to b} = \frac{\pi}{3\varepsilon_0 \hbar^2} \left| \vec{\mathcal{P}}_{ab} \right|^2 \rho(\omega_0)$$

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$\mathcal{P}^x_{32}$ $\qquad$ $\mathcal{P}^y_{32}$

A collection of 100,000 of these molecules are prepared in the $|\psi_2\rangle$ state. The molecules are in an environment where the background radiation density is as shown in the plot.

Q: What is $|\vec{\mathcal{P}}_{ab}|^2$ for $R_{2\to 3}$?

① P_0^2 ② $2P_0^2$ ③ $3P_0^2$ ④ $4P_0^2$ ⑤ $5P_0^2$

$$|\vec{\mathcal{P}}_{ab}|^2 = \mathcal{P}^x_{32}\cdot(\mathcal{P}^x_{32})^* + \mathcal{P}^y_{32}\cdot(\mathcal{P}^y_{32})^* + \mathcal{P}^z_{32}\cdot(\mathcal{P}^z_{32})^*$$

$$= P_0^2(1^2 + 2^2) = 5P_0^2$$

$$R_{a\to b} = \frac{\pi}{3\varepsilon_0 \hbar^2} \left| \vec{\mathcal{P}}_{ab} \right|^2 \rho(\omega_0)$$

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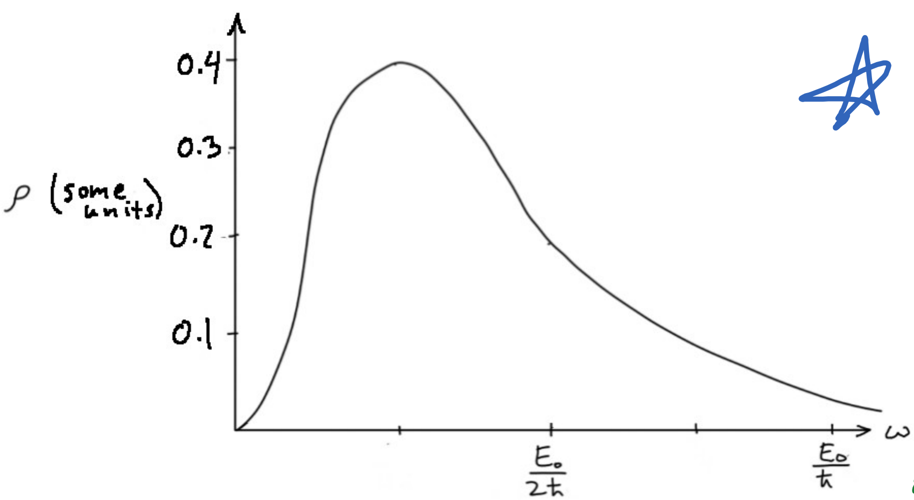


★ What is the expected amount of time before 10 molecules are in the state $|\psi_3\rangle$? You can ignore processes with multiple transitions (e.g. 2 -> 1 -> 3).

Q: After plugging in numbers, we find

$$R_{2\to 3} = 2\times 10^3\, s^{-1}$$

What is the answer to ★?

R = probability per unit time of making transition

(see next slide)

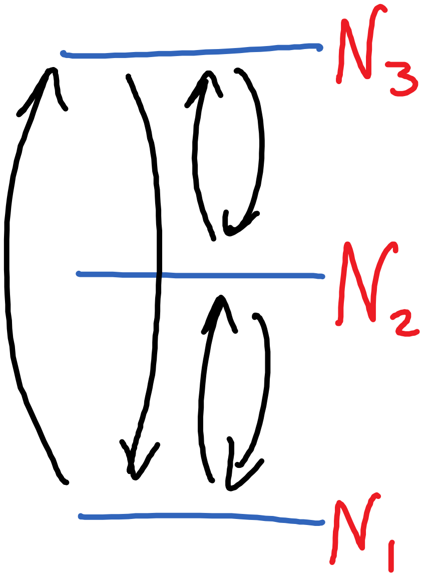
$R_{2\to 3}$: probability per unit time of making a transition.

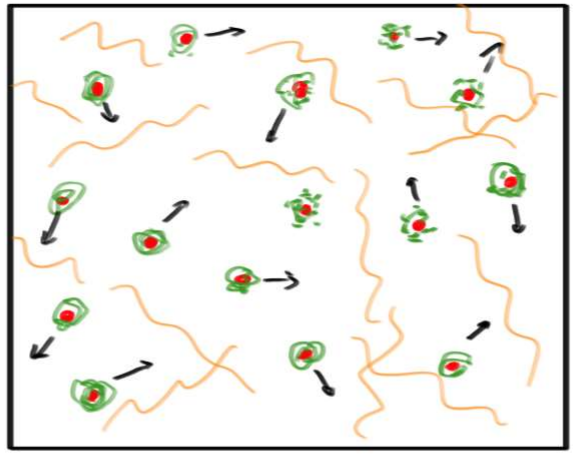
In time dt, each atom in the state $|\psi_2\rangle$ will have made transition to $|\psi_3\rangle$ with probability $R\,dt$

With N atoms in state $|\psi_2\rangle$: expected number to transition to state $|\psi_3\rangle$ in time dt is: $N_{2\to 3} = N \times (R\,dt)$

Want: $10 = 100{,}000 \times (2\times 10^3 s^{-1}) \times dt$

$$\Rightarrow dt = 5\times 10^{-8} s$$

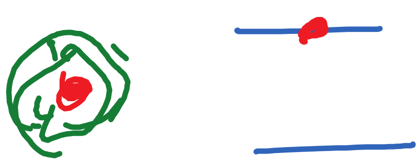




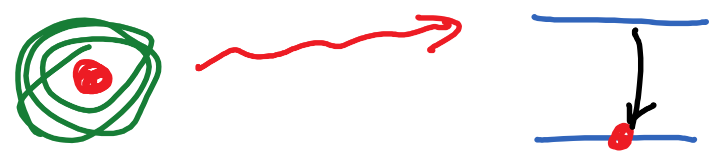
Generally: can have transitions to higher energy states through absorption, and transitions to lower energy states through stimulated emission and SPONTANEOUS EMISSION

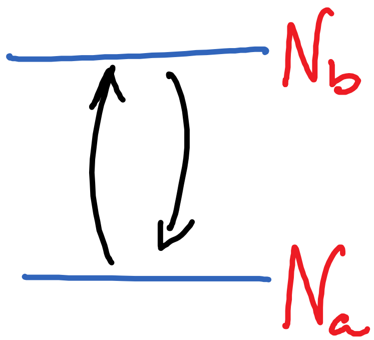
spontaneous emission:

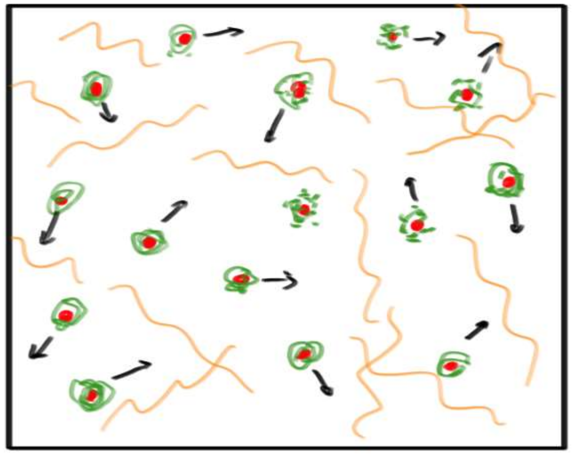
BEFORE:



AFTER:







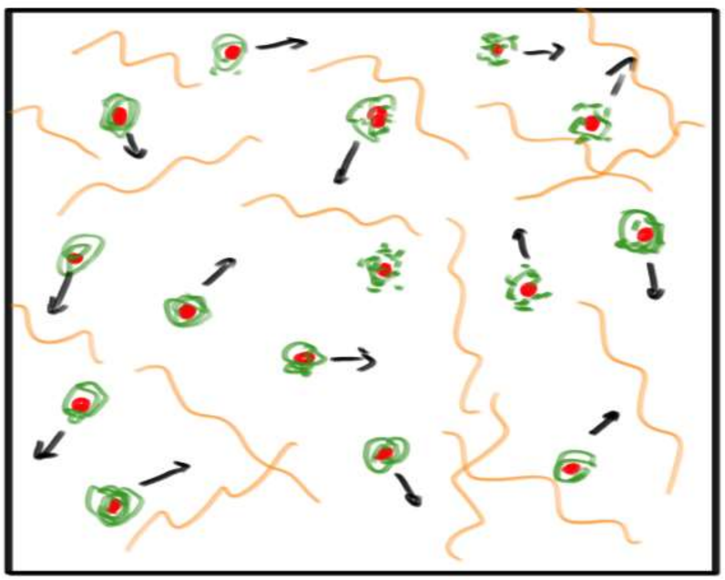
Q: suppose we have absorption rate $R_{a \to b}$, stimulated emission rate $R_{b \to a}$, and spontaneous emission rate A.

What is $\dfrac{dN_b}{dt}$ in terms of N_a, N_b, A, $R_{a \to b}$, and $R_{b \to a}$?

How is $\dfrac{dN_a}{dt}$ related to this?

$$\frac{dN_b}{dt} = -N_b A - N_b R_{b \to a} + N_a R_{a \to b}$$

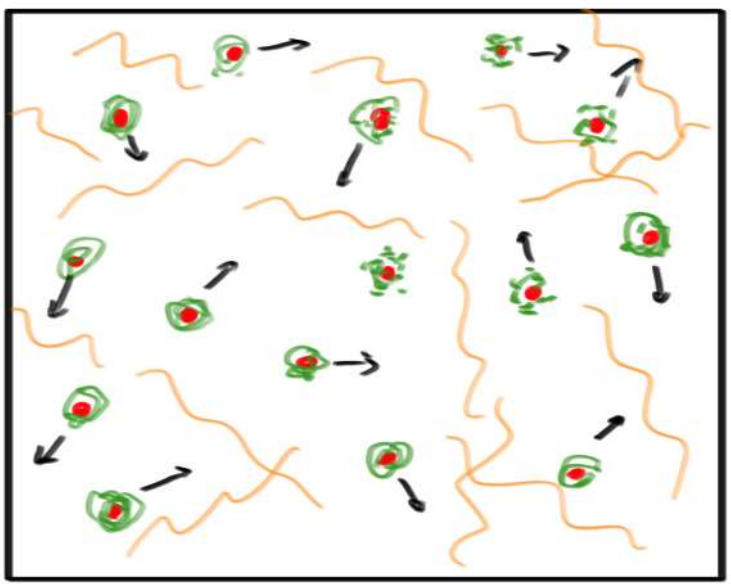
INTERLUDE: Thermal equilibrium



Q: What are $\frac{dN_a}{dt}$ and $\frac{dN_b}{dt}$ in thermal equilibrium?

INTERLUDE: Thermal equilibrium ← nothing is changing

(1) $$\frac{dN_a}{dt} = \frac{dN_b}{dt} = 0$$



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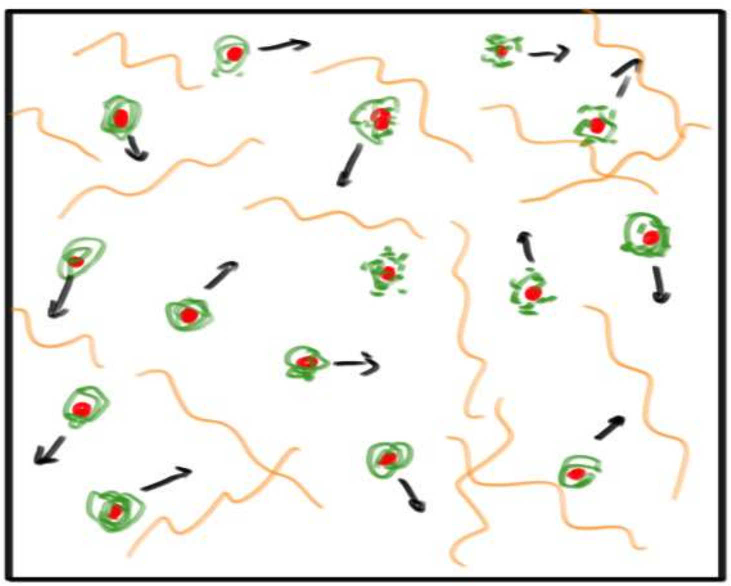
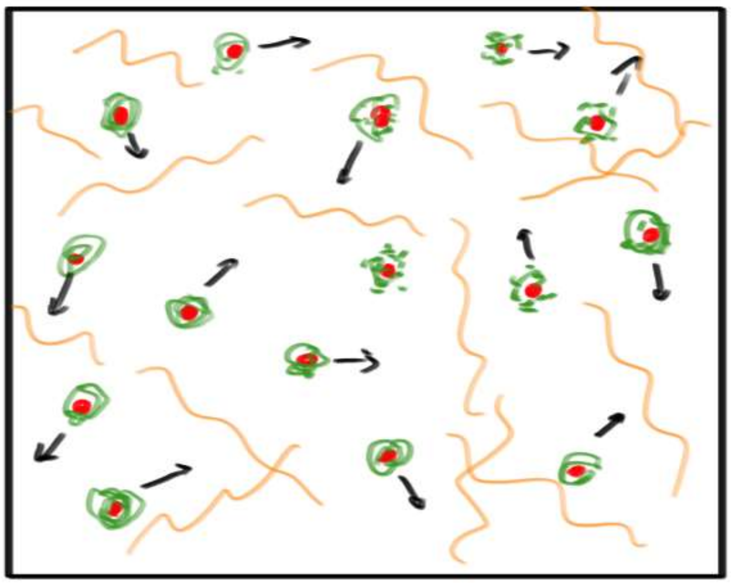


① $$\frac{dN_a}{dt} = \frac{dN_b}{dt} = 0$$

② $$N_i \propto e^{-E_i/kT}$$

"Boltzmann Factor"

INTERLUDE: Thermal equilibrium



① $$\frac{dN_a}{dt} = \frac{dN_b}{dt} = 0$$

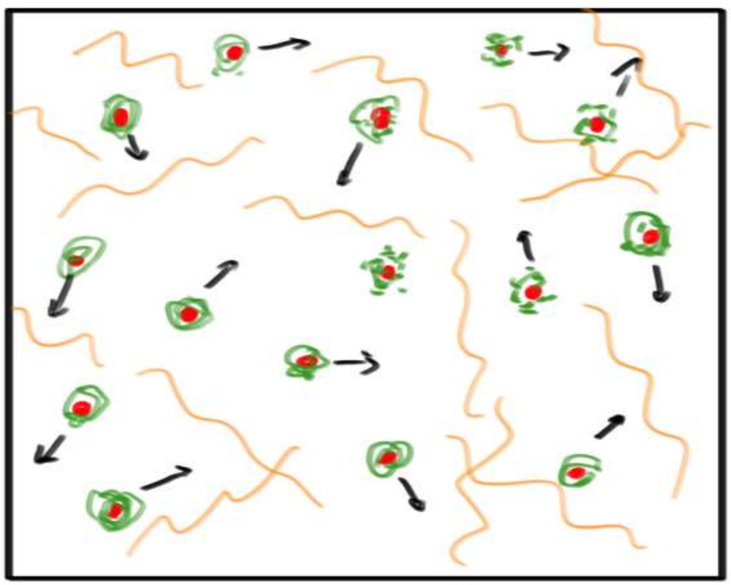
② $$N_i = \mathcal{C} e^{-E_i/kT}$$

some constant.

③ $$\rho(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/kT} - 1}$$

Planck formula

INTERLUDE: Thermal equilibrium



① $$\frac{dN_a}{dt} = \frac{dN_b}{dt} = 0$$

② $$N_i = C e^{-E_i/kT}$$

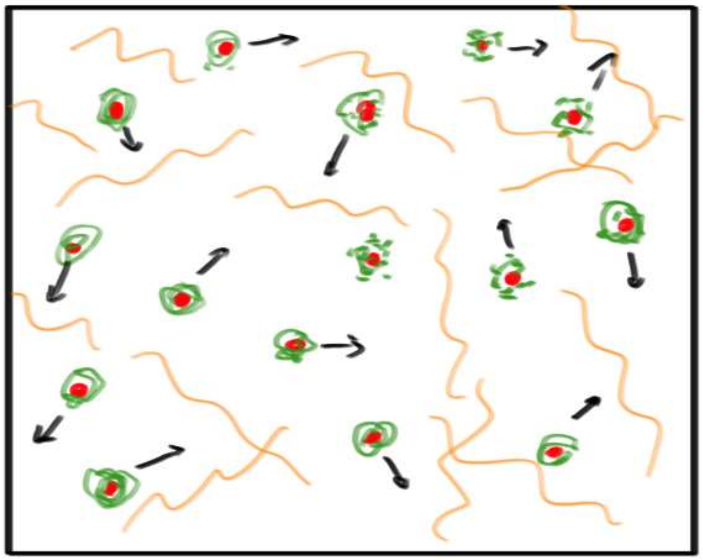
③ $$\rho(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/kT} - 1}$$

We derived: $$\frac{dN_b}{dt} = -N_b \cdot A - N_b \cdot R_{b\to a} + N_b \cdot R_{a\to b} \qquad ⭑$$

Also: $$R_{a\to b} = R_{b\to a} = \frac{\pi}{3\varepsilon_0 \hbar^2} |\vec{p}_{ab}|^2 \cdot \rho(\omega_{ba})$$

Use ⭑ to solve for A

INTERLUDE: Thermal equilibrium



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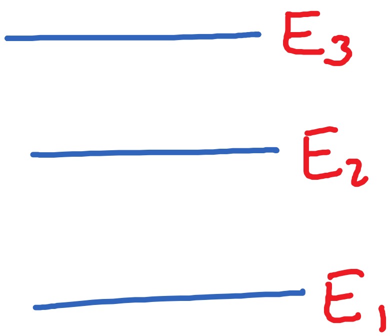
Also: $$R_{a\to b} = R_{b\to a} = \frac{\pi}{3\varepsilon_0 \hbar^2} |\vec{\wp}_{ab}|^2 \cdot \rho(\omega_{ba})$$

Use ⊛ to solve for A. Result: $$\boxed{A = \frac{\omega_{ba}^3 \, |\vec{\wp}_{ab}|^2}{3\pi \varepsilon_0 \hbar c^3}}$$



Q: A system starts with 1000 atoms in state $|E_3\rangle$. If $A_{3\to2}$ and $A_{3\to1}$ are the spontaneous transition rates for this system, how many atoms are expected to remain in the state $|E_3\rangle$ after time t?

$$\frac{dN_3}{dt} = -N_3 A_{3\to2} - N_3 A_{3\to1}$$

$$\frac{dN_3}{dt} = -\left(A_{3\to2} + A_{3\to1}\right) N_3$$

$$N_3 = e^{-\left(A_{3\to2} + A_{3\to1}\right)t} \cdot N_3(0)$$