Perturbation Theory Worksheet

1. Suppose we have a system with energy eigenstates $|E=0\rangle$, $|E=E_0\rangle$, $|E=3E_0\rangle$, so the Hamiltonian in this basis is:

$$H_0 = E_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

We now add a perturbation λH_1 where H_1, written in the energy basis is

$$H_1 = E_0 \begin{pmatrix} 0 & 4 & 2 \\ 4 & 3 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

a) What are the energies of the three eigenstates of $H_0 + \lambda H_1$ to first order in λ?

$$E_1 = 0 + \lambda \cdot \langle E=0|H_1|E=0\rangle = 0 + \lambda \cdot 0 = 0$$

$$E_2 = E_0 + \lambda \langle E=E_0|H_1|E=E_0\rangle = E_0 + \lambda \cdot 3E_0$$

$$E_3 = 3E_0 + \lambda \langle E=3E_0|H_1|E=3E_0\rangle = 3E_0 - \lambda E_0$$

b) Write an expression for the ground state of the perturbed system, to first order in λ, using the original energy basis.

$$|E=0\rangle + \lambda \cdot |E=E_0\rangle \frac{\langle E=E_0|H_1|E=0\rangle}{0 - E_0} + \lambda |E=3E_0\rangle \frac{\langle 3E_0|H_1|0\rangle}{0-3E_0}$$

$$= |E=0\rangle - 4\lambda |E=E_0\rangle - \frac{2}{3}\lambda |E=3E_0\rangle$$

c) What is the ground state energy at second order in λ?

$$\lambda^2 E_1^{(2)} = \lambda^2 \cdot \frac{|\langle E_0 | H_1 | 0\rangle|^2}{0 - E_0} + \lambda^2 \frac{|\langle 3E_0 | H_1 | 0\rangle|^2}{0 - 3E_0}$$

$$= \lambda^2 \cdot \left(\frac{16 E_0^2}{-E_0}\right) + \lambda^2 \left(\frac{4E_0^2}{-3E_0}\right) = -\frac{52}{3} E_0 \cdot \lambda^2$$

② a) Consider the quantum Harmonic Oscillator, with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

This has energy eigenstates $|n\rangle$ with energies $E_n = \hbar\omega(n + \frac{1}{2})$. If we add a perturbation

$$H \to H + \lambda x^4$$

what is the change in the energy of the ground state to first order in λ. At least write what you need to calculate and say how you would do it.

$$\delta E = \lambda \cdot \langle 0 | x^4 | 0 \rangle$$

$$= \lambda \cdot \left(\frac{\hbar}{2m\omega}\right)^2 \langle 0 | (a + a^\dagger)^4 | 0\rangle$$

norm of $(a+a^\dagger)(a+a^\dagger)|0\rangle$

$= (a + a^\dagger)|1\rangle = |0\rangle + \sqrt{2}\,|2\rangle$

norm is $1^2 + \sqrt{2}^2 = 3$

$$\delta E = \frac{3\lambda\hbar^2}{4m^2\omega^2}$$

b) BONUS: How does the fractional shift $\frac{\delta E_n}{E_n^0}$ depend on n for a harmonic oscillator? What does this suggest about the result for large n?