Name: SOLUTIONS:

Physics 200 Tutorial 10:

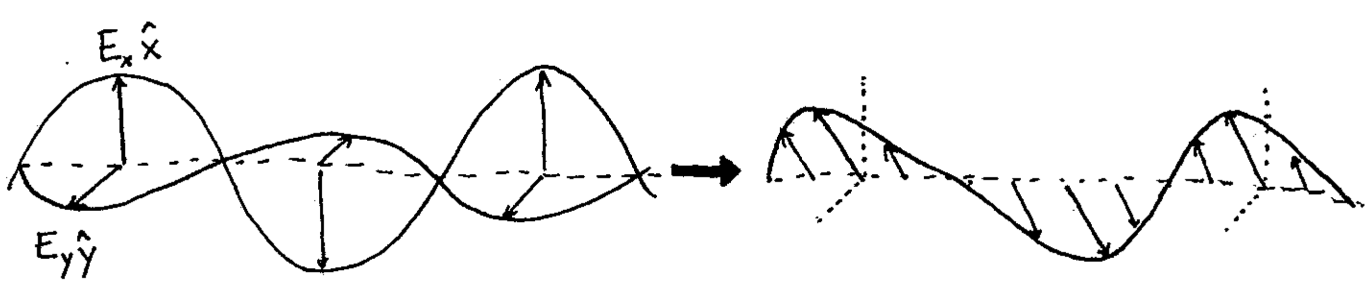
Complex Numbers and Quantum Superposition

By thinking about the photon picture of polarizer experiments, we have been led to the idea of quantum superposition. An important feature of this is that if $|a\rangle$ and $|b\rangle$ are two states of a physical system (perhaps with definite values for some physical property such as position) then we can also have a state

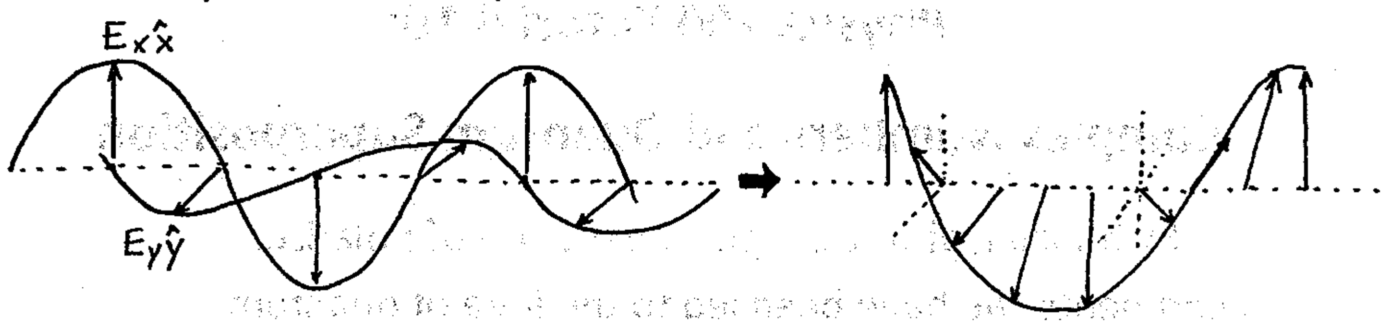
$$\alpha\,|a\rangle + \beta\,|b\rangle$$

Up until now, we have been assuming that α and β are real numbers, but today we will see that in order to describe the most general states, we need to allow α and β to be *complex numbers.*

Our starting point will be the model we have developed for polarized light. Classically, we say that light polarized along any direction could be obtained by superposing light polarized along x and light polarized along y (assuming that the light is travelling in the z direction). But so far, we have only considered adding these two components in phase:

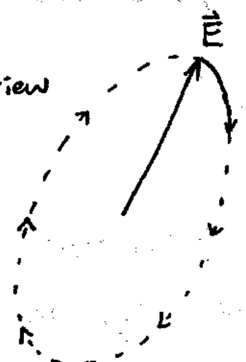


But we can get more general polarizations of light by adding the two components out of phase:



Now instead of oscillating back and forth in one direction (so called LINEAR POLARIZATION), the electric field rotates around in an ellipse. This is known as ELLIPTICAL POLARIZATION.

Head-on view of $\vec{E}$ at a fixed location.



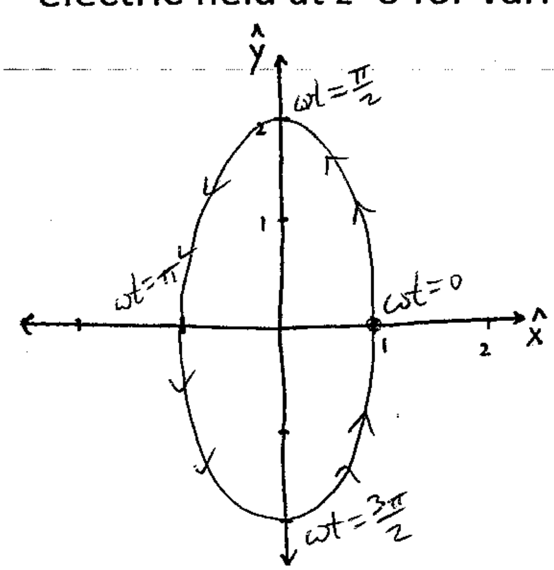
Mathematically, we can write the electric field as

$$\vec{E} = E_x\,\hat{x}\cos(k\,z - \omega\,t + \phi_x) + E_y\,\hat{y}\cos(k\,z - \omega\,t + \phi_y)$$

where $k = 2\pi/\lambda$ and $\omega = 2\pi f$. For $\phi_x = \phi_y$, we just have ordinary (linear) polarization, but for $\phi_x \neq \phi_y$, we have the more general case of elliptically polarized light.

Question 1

As an example, let $E_x = 1$, $E_y = 2$, $\phi_x = 0$ and $\phi_y = \pi/2$. Plot the electric field at z=0 for various values of ωt between 0 and 2π.



$\vec{E} = \hat{x}\cos(-\omega t) + 2\hat{y}\cos(-\omega t + \frac{\pi}{2})$

$\omega t = 0 \qquad \vec{E} = \hat{x}$

$\omega t = \frac{\pi}{2} \qquad \vec{E} = 2\hat{y}$

$\vdots$

Now, suppose we have a photon of this elliptically polarized light. How can we represent this in our mathematical model where photon states were unit vectors? Before, we said that any state could be written as

$$\alpha\,|0°\rangle + \beta\,|90°\rangle \qquad |\alpha|^2 + |\beta|^2 = 1$$

But to represent an elliptically polarized photon, we somehow want to add up these basis vectors "out of phase". We will see that the natural way to do this is to let α and β be complex numbers. But first, we'd better review some things about complex numbers.

Question 2

a) You probably know that complex numbers are numbers that we can write as

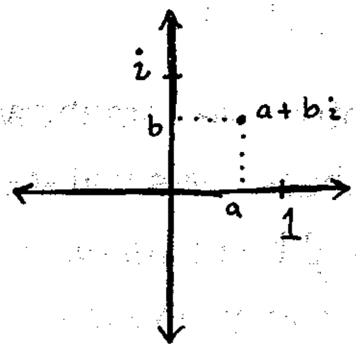
$$z = a + b\,i$$

where a and b are real numbers and i is some magical number with $i \times i = -1$. This is enough information to add and multiple any two complex numbers. As an example, calculate the following:

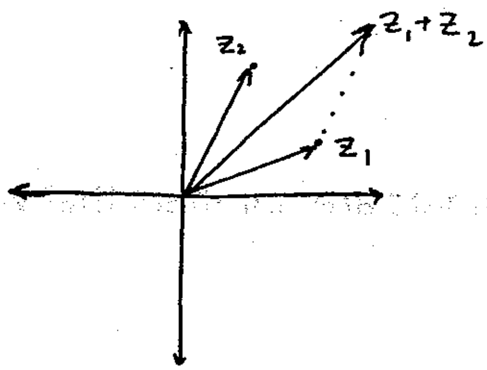
$$(3 + 2\,i) + (4 + 7\,i) \;=\; 7 + 9\,i$$

$$(\sqrt{3} + i) \times (1 + \sqrt{3}\,i) \;=\; 4\,i$$

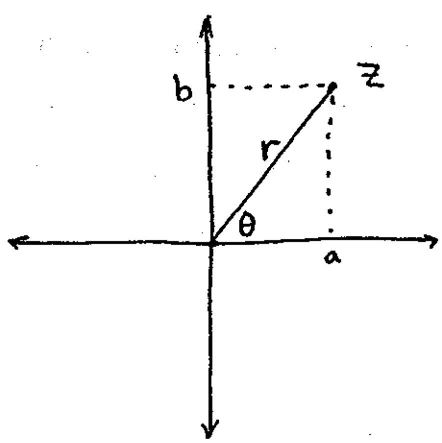
In order to visualize complex numbers, it really helps to think of them as points in a 2D plane, where the number 1 is at distance 1 along the horizontal axis and the number i is at distance 1 along the vertical axis (figure 1).



Then adding complex numbers is just adding the vectors (figure 2). To understand how to visualize multiplication, it is easiest to think in terms of "polar coordinates."



In figure 3, we see that any complex number can also be described by giving its MAGNITUDE r (the length from 0 to z, also knowns as the MODULUS), and its PHASE (the angle θ between the vector and the "real axis"). The product of two complex numbers with polar coordinates (r_1, θ_1) and (r_2, θ_2) is a complex number with polar coordinates $(r_1 \cdot r_2, \theta_1 + \theta_2)$.



In other words, to multiply two complex numbers represented in polar coordinates, we just multiply the magnitudes to get the new magnitude and add the phases to get the new phase.

b) Exercise: suppose that $z_1 = 1 + \sqrt{3}\,i$ and $z_1 = \sqrt{3} + i$. Then:

$r_1 = 2$

$\theta_1 = \frac{\pi}{3}$

$r_2 = 2$

$\theta_2 = \frac{\pi}{6}$

$r_1 r_2 = 4$

$\theta_1 + \theta_2 = \frac{\pi}{2}$

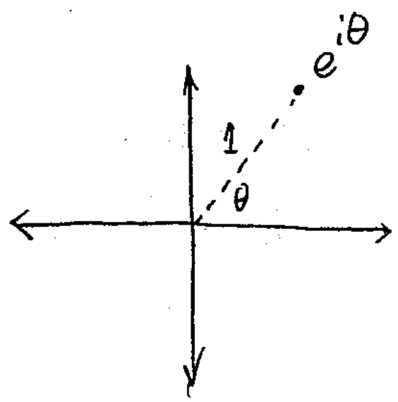
You already calculated $(1 + \sqrt{3}\,i) \times (\sqrt{3} + i) = z_3$ in part a. For this number, what are r_3 and θ_3?

$r_3 = 4$

$\theta_3 = \frac{\pi}{2}$

Do you find $r_3 = r_1 r_2$ and $\theta_3 = \theta_1 + \theta_2$? YES

c) A very important fact about complex numbers is that $e^{i\theta}$ is a complex number with magnitude 1 and phase θ. We can show this by writing



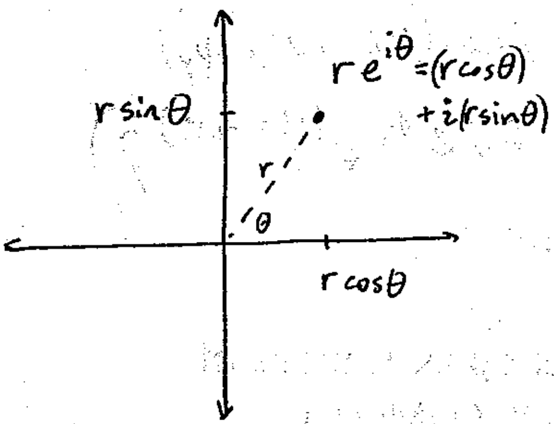
$$e^{i\theta} = 1 + i\theta + (i\theta)^2/2 + (i\theta)^3/3! + \ldots$$

$$= (1 - \theta^2/2 + \ldots) + i\,(\theta - \theta^3/3! + \ldots)$$

$$= \cos\theta + i\sin\theta$$

Here, we have used the Taylor expansions of e^x, cos(x), and sin(x).

This means that a complex number with magnitude r and phase θ can be written as $z = r\,e^{i\theta}$.



As an example, what are the real and imaginary parts of $3\,e^{i\cancel{\theta}\pi}$?

$$3\,e^{i\cancel{\theta}\pi} = (-3) + i\,(0)$$

Question 3

BACK TO PHYSICS...

Now suppose we have a wave with some general phase and amplitude:

$$h = A\cos(k\,x - \omega\,t + \phi)$$

We can use the relation $r\,e^{i\theta} = (r\cos\theta) + i\,(r\sin\theta)$ to say that

real part

$$h = Re(\,A\,e^{i(k\,x-\omega\,t+\phi)}\,)$$

$$= Re(\,A\,e^{i\phi}\;e^{i(k\,x-\omega\,t)}\,)$$

$$= Re(\,Z\,e^{i(k\,x-\omega\,t)}\,)$$

where we have defined $Z = A\,e^{i\phi}$. So the information about the amplitude and phase of the wave is completely contained in the complex number Z.

This way of representing things is extremely useful when it comes to adding up waves that are out of phase. For example, if we have

$$h = A_1 \cos(k\,x - \omega\,t + \phi_1) + A_2 \cos(k\,x - \omega\,t + \phi_2)$$

it's not obvious how to find the amplitude of the resulting wave. But using the complex number representation, we have:

$$\begin{aligned} h &= Re(\, Z_1\, e^{i(k\,x-\omega\,t)} + Z_2\, e^{i(k\,x-\omega\,t)}) \\ &= Re(\,(Z_1 + Z_2\,)\, e^{i(k\,x-\omega\,t)}) \end{aligned}$$

where $Z_1 = A_1\, e^{i\phi_1}$ and $Z_2 = A_2\, e^{i\phi_2}$. So the amplitude of the resulting wave is just the magnitude of $Z_1 + Z_2$ and the phase is the phase of $Z_1 + Z_2$.

Example: we can write the sum of two waves

$$2\cos(k\,x - \omega\,t) + \cos(k\,x - \omega\,t + \pi/3)$$

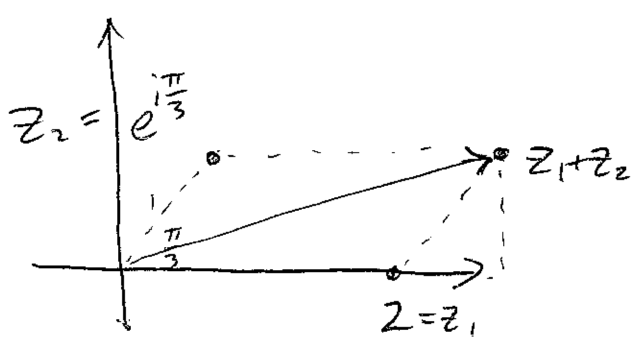
as $A \cos(k\,x - \omega\,t + \phi)$. What are A and ϕ?

$$= Re(\underbrace{2}_{Z_1}\, e^{i(kx-\omega t)} + \underbrace{e^{i\frac{\pi}{3}}}_{Z_2}\, e^{i(kx-\omega t)})$$

$$A = \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{7}$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{3}}{5}\right)$$

$$\begin{aligned} Z_1 + Z_2 &= 2 + e^{i\frac{\pi}{3}} \\ &= \left(2 + \cos\left(\tfrac{\pi}{3}\right)\right) + i\left(\sin\left(\tfrac{\pi}{3}\right)\right) \\ &= \frac{5}{2} + i\,\frac{\sqrt{3}}{2} \end{aligned}$$



It's now easy to see why complex numbers will be useful in representing elliptically polarized light. We just notice that

$$E_x \hat{x} \cos(kz - \omega t + \phi_x) + E_y \hat{y} \cos(kz - \omega t + \phi_y)$$
$$= Re\left(E_x e^{i\phi_x} e^{i(kz-\omega t)} \hat{x} + E_y e^{i\phi_y} e^{i(kz-\omega t)} \hat{y} \right)$$
$$= Re\left((Z_x \hat{x} + Z_y \hat{y}) e^{ikz-\omega t} \right)$$

So that in the classical description, different polarizations of light are in one-to-one correspondence with COMPLEX SUPERPOSITIONS $Z_x \hat{x} + Z_y \hat{y}$. We can't really draw such a vector when Z_x and Z_y are not real, but the important thing is that the information about the amplitudes and phases of the two different components of the light are contained in the complex numbers Z_x and Z_y.

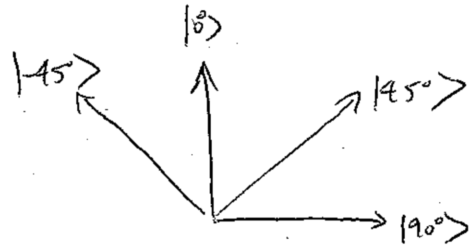
So how do we represent a photon of this elliptically polarized light in our mathematical model? We just allow complex superpositions of the eigenstates:

$$z_1 |0^\circ\rangle + z_2 |90^\circ\rangle \qquad \text{where } |z_1|^2 + |z_2|^2 = 1$$

When z_1 and z_2 have different phases, the state describes a photon of elliptically polarized light. If the light is incident on a 0° polarizer, we still have the rule that the photon will pass through with probability $|z_1|^2$, but now the $|z_1|$ represents the magnitude of the complex number z_1.

see next page.

Exercise: A photon in a state $\frac{1}{2}|0^\circ\rangle + i\frac{\sqrt{3}}{2}|90^\circ\rangle$ is incident on a 45° polarizer. What is the probability that it will go through? (hint: first write $|0^\circ\rangle$ and $|90^\circ\rangle$ in terms of the eigenstates of the 45° polarizer).



$$|0^\circ\rangle = \frac{1}{\sqrt{2}}|45^\circ\rangle + \frac{1}{\sqrt{2}}|-45^\circ\rangle$$

$$|90^\circ\rangle = \frac{1}{\sqrt{2}}|45^\circ\rangle - \frac{1}{\sqrt{2}}|-45^\circ\rangle$$

$$\frac{1}{2}|0^\circ\rangle + i\frac{\sqrt{3}}{2}|90^\circ\rangle = \left(\frac{\sqrt{2}}{4} + i\frac{\sqrt{6}}{4}\right)|45^\circ\rangle + \left(\frac{\sqrt{2}}{4} - i\frac{\sqrt{6}}{4}\right)|45^\circ\rangle$$

$$\text{Prob} = \left|\frac{\sqrt{2}}{4} + i\frac{\sqrt{6}}{4}\right|^2$$

$$= \frac{2}{16} + \frac{6}{16}$$

$$= \frac{8}{16}$$

$$\Rightarrow \boxed{P = \frac{1}{2}}$$

For a photon of light in the example of Question 1, what are z_1 and z_2?

for classical wave:

$$z_1 = 1$$

$$z_2 = 2e^{i\frac{\pi}{2}} = 2i$$

for photon, need $|z_1|^2 + |z_2|^2 = 1$ $\therefore$ divide by $\sqrt{5}$ to normalize:

$$\boxed{z_1 = \frac{1}{\sqrt{5}} \qquad z_2 = \frac{2i}{\sqrt{5}}}$$