Name & Student number: SOLUTIONS

Physics 402 Midterm, March 5th, 2020

Please read and observe the following rules for this midterm

1. No smoking.
2. No spitting.
3. No public nudity.
4. The use of chewing tobacco during the midterm is strictly prohibited (see also #2).
5. Profanity should be used only if absolutely necessary.
6. Wild animals are not allowed in the examination room, with the exception of koalas.
7. Costumes may be worn, provided that no more than two people are dressed as Billie Eilish.
8. Calculators are permitted, provided that you do not try to spell swear words with upside-down numbers (see #5).
9. The use of Greek letters is permitted only during the first 40 minutes and the last 40 minutes of this examination.
10. The writing of additional rules below the rules in this list is strictly forbidden, unless those additional rules are properly numbered.

Multiple Choice Questions (one point each):
Please write your answers in the spaces on page 4

1) For a quantum system with a two-dimensional Hilbert space, the set of distinct quantum states can be represented geometrically as

a) a line

b) a circle

c) a plane

d) a two-dimensional sphere

e) four-dimensional Euclidean space

distinct states can be represented as
$\cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2} e^{i\varphi}|\downarrow\rangle$
for $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$

2) If $|\Psi\rangle$ is a normalized state of a quantum system, and $|\lambda\rangle$ is a normalized eigenstate of some physical observable, we can say that

$|\langle\lambda|\Psi\rangle|^2$ is probability of finding λ in measurement, so $|\langle\lambda|\Psi\rangle| \le 1$

a) $\langle\lambda|\Psi\rangle$ is always real and positive.

b) $\langle\lambda|\Psi\rangle$ is always real but can be positive or negative.

c) $\langle\lambda|\Psi\rangle$ can be complex but must have norm less than or equal to 1.

c) $\langle\lambda|\Psi\rangle$ can be any complex number.

3) For a quantum system with a dimension 2 Hilbert space, we can say that

a) all states must have the same energy.

b) there can be at most two possible outcomes in a measurement of energy, regardless of which state we measure.

can pick basis of energy eigenstates $\rightarrow$ 2 basis vectors $\rightarrow$ possible energies are the energies of these

c) a measurement of energy can yield infinitely many possible values depending on the state, but these will always be in some finite range $[E_1, E_2]$.

d) there exist states for which a measurement of energy can yield any positive value.

4) A quantum system consists of three qubits. For this system, the dimension of the Hilbert space (i.e. the number of independent basis vectors) is

Basis vectors $|\uparrow\uparrow\uparrow\rangle, |\uparrow\uparrow\downarrow\rangle$ etc... 8 total.

a) 3 b) 4 c) 6 d) 8 e) 14

5) In a qubit system with basis states $|\uparrow\rangle$ and $|\downarrow\rangle$, the operator corresponding to some physical observable $\mathcal{O}$ acts on $|\uparrow\rangle$ as $\hat{O}|\uparrow\rangle = 2|\uparrow\rangle - |\downarrow\rangle$. Which of the following is possible for the action of $\hat{O}$ on $|\downarrow\rangle$?

Matrix for $\mathcal{O}$ must be Hermitian, so

$$\langle\downarrow|\hat{O}|\uparrow\rangle = \langle\uparrow|\hat{O}|\downarrow\rangle^*$$

a) $\hat{O}|\downarrow\rangle = 2|\uparrow\rangle - |\downarrow\rangle$

b) $\hat{O}|\downarrow\rangle = -|\uparrow\rangle + 3|\downarrow\rangle$

c) $\hat{O}|\downarrow\rangle = 5|\uparrow\rangle$

answer b) gives $\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$ ✓

d) $\hat{O}|\downarrow\rangle = |\uparrow\rangle + 2|\downarrow\rangle$

6) For a state $|\Psi\rangle$ of a particle in one-dimension with wavefunction $\psi(x)$. Define another state $|\chi\rangle = (\mathbf{1} - i\epsilon\hat{P}/\hbar)|\Psi\rangle$ where $\hat{P}$ is the momentum operator. For infinitesimal ϵ, we can say that the wavefunction for $|\chi\rangle$ is

infinitesimal translation operator: moves wavefn to right by ε

a) $\psi(x) - \epsilon$

b) $\psi(x - \epsilon)$

c) $-\epsilon\psi'(x)$

7) If $|\Psi(t)\rangle$ solves the (time-dependent) Schrödinger equation for some time-independent Hamiltonian H, we can say that

a) the state $|\Psi(t)\rangle$ has a definite energy, and this energy does not change with time.

b) the state $|\Psi(t)\rangle$ has a definite energy at any time, but this energy can change with time.

c) the state $|\Psi(t)\rangle$ does not necessarily have a definite energy, but the expectation value of energy does not change with time.

→ any initial state is possible

→ energy is conserved since H time-indep.

d) the state $|\Psi(t)\rangle$ does not necessarily have a definite energy, and the expectation value of energy can change with time.

8) If the state of a quantum system with a time-independent Hamiltonian is some energy eigenstate $|E\rangle$ at time $t = 0$, which of the following is necessarily true?

a) Physical observables for this system will oscillate periodically with a frequency proportional to the energy E.

$$\langle\Psi(t)|\hat{O}|\Psi(t)\rangle = \langle\Psi(0)|e^{iEt/\hbar}\hat{O}e^{-iEt/\hbar}|\Psi(0)\rangle$$

b) All physical observables will be independent of time for this state.

$$= \langle\Psi(0)|\hat{O}|\Psi(0)\rangle$$

c) The state has a definite value for all physical observables.

d) The state has a definite value for any physical observable associated with a time-independent operator.

9) A certain source produces spin half particles in identical spin states. In order to determine this spin state (to high accuracy), the minimal thing we need is

a) a single measurement of any component of the spin.

b) a single measurement of some specific component of the spin.

c) single measurements of several different components of the spin.

d) repeated measurements of any component of the spin.

e) repeated measurements of multiple different components of spin.

recall our in-class activity..

10) Suppose we add a perturbation $H_1 = \lambda x^3$ to a harmonic oscillator. The first non-zero correction to the energy of the ground state comes at second order in perturbation theory. In the sum over states appearing in the formula for the second order energy shift, how many terms are non-zero in this case?

$$\delta E_2 = \lambda^2 \sum_{n \neq 0} \frac{|\langle n | H_1 | 0 \rangle|^2}{E_0 - E_n}$$

a) 1

b) 2

c) 3

d) 4

e) an infinite number

$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$ so $x^3|0\rangle$ is a linear comb. of $|1\rangle, |3\rangle$

3
2
1
0

Write your multiple choice answers here:

1	2	3	4	5
6	7	8	9	10

Long Answer Question 1 (6 points)

A qubit system with basis states $|\uparrow\rangle$ and $|\downarrow\rangle$ has a Hamiltonian represented in this basis by

$$H_0 = E\begin{pmatrix} 6 & 2i \\ -2i & 3 \end{pmatrix} \tag{1}$$

where $|\uparrow\rangle$ is represented as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle$ is represented as $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

a) Write the energy eigenstates for this system in the form $a|\uparrow\rangle + b|\downarrow\rangle$, and give the energy eigenvalue for each.

b) What is the expectation value for a measurement of S_z in the ground state of this system?

c) If we add a perturbation of the form

$$H_1 = \frac{1}{50}E\begin{pmatrix} 2 & 5 \\ 5 & 0 \end{pmatrix} \tag{2}$$

to the Hamiltonian, by what percentage does the ground state energy change, in the approximation of first-order perturbation theory?

a) Eigenvalues of H_0 are E times eigevalues of $\begin{pmatrix} 6 & 2i \\ -2i & 3 \end{pmatrix}$. Char. polynomial is

$$\lambda^2 - 9\lambda + 14 = 0 \Rightarrow (\lambda-2)(\lambda-7) = 0 \rightarrow \lambda = 2, 7$$

$\lambda = 2$: eigenvector is $\begin{pmatrix} 2i \\ -4 \end{pmatrix} \xrightarrow{\text{normalize}} \frac{1}{\sqrt{20}}\begin{pmatrix} 2i \\ -4 \end{pmatrix} = \begin{pmatrix} i/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}$

State $\frac{i}{\sqrt{5}}|\uparrow\rangle - \frac{2}{\sqrt{5}}|\downarrow\rangle$ has energy $2E$

$\lambda = 7$ eigenvector is $\begin{pmatrix} 2i \\ 1 \end{pmatrix} \xrightarrow{\text{normalize}} \frac{1}{\sqrt{5}}\begin{pmatrix} 2i \\ 1 \end{pmatrix}$

State $\frac{2i}{\sqrt{5}}|\uparrow\rangle + \frac{1}{\sqrt{5}}|\downarrow\rangle$ has energy $7E$

b) We have $\langle S_z\rangle = P_\uparrow \cdot \frac{\hbar}{2} + P_\downarrow\left(-\frac{\hbar}{2}\right)$. Ground state is energy $2E$ state

so $P_\uparrow = \left|\frac{i}{\sqrt{5}}\right|^2 = \frac{1}{5}$, $P_\downarrow = \left|-\frac{2}{\sqrt{5}}\right|^2 = \frac{4}{5}$

$$\langle S_z\rangle = \frac{1}{5}\cdot\frac{\hbar}{2} + \frac{4}{5}\left(-\frac{\hbar}{2}\right) = -\frac{3\hbar}{10}$$

c) We have that

$$
\begin{aligned}
\delta E &= \langle 2E | H_1 | 2E \rangle \\
&= \begin{pmatrix} -\frac{i}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} \cdot \frac{1}{50} E \begin{pmatrix} 2 & 5 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} \frac{i}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} \\
&= \frac{E}{50} \cdot \begin{pmatrix} -\frac{i}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} -\frac{10}{\sqrt{5}} + \frac{2i}{\sqrt{5}} \\ \frac{5i}{\sqrt{5}} \end{pmatrix} \\
&= \frac{E}{50} \cdot \left(\frac{10i}{5} + \frac{2}{5} - \frac{10i}{5} \right) \\
&= \frac{E}{125}
\end{aligned}
$$

So the ground state energy increases by a fraction $\frac{E/125}{2E} = 0.4\%$

Long Answer Question 2 (6 points)

A certain kind of particle in one dimension has Hamiltonian

$$H_0 = E_0 a^\dagger a^\dagger a a \tag{3}$$

where a and $a^\dagger$ are the creation and annihilation operators for a harmonic oscillator with mass m and frequency ω.

a) Show that the usual harmonic oscillator eigenstates $|n\rangle$ are energy eigenstates for this Hamiltonian, and calculate the energy of the state $|n\rangle$.

b) For an initial state $\frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle$, what is $\langle \Psi(t)|x|\Psi(t)\rangle$?

c) If we now add a perturbation

$$\lambda H_1 = \lambda E_0 (a + a^\dagger) \tag{4}$$

to the Hamiltonian, calculate the energies of the two lowest energy states to first order in λ.

a) We have that
$$\begin{aligned} H_0|n\rangle &= E_0 a^\dagger a^\dagger a a |n\rangle \\ &= E_0 a^\dagger a^\dagger a \sqrt{n}|n-1\rangle \\ &= E_0 a^\dagger a^\dagger \sqrt{n}\cdot\sqrt{n-1}\,|n-2\rangle \\ &= E_0 a^\dagger \sqrt{n}\cdot\sqrt{n-1}\cdot\sqrt{n-1}\,|n-1\rangle \\ &= E_0 \sqrt{n}\cdot\sqrt{n-1}\cdot\sqrt{n-1}\cdot\sqrt{n}\,|n\rangle \\ &= E_0\cdot n\cdot(n-1)|n\rangle \end{aligned}$$

So $|n\rangle$ is an energy eigenstate w. eigenvalue $E_0 \cdot n(n-1)$

c) In the $\lambda = 0$ system, states $n=0$ and $n=1$ both have energy 0, so we need to use degenerate perturbation theory. We have:

$$H_1|0\rangle = E_0(a+a^\dagger)|0\rangle = E_0|1\rangle$$
$$H_1|1\rangle = E_0(a+a^\dagger)|1\rangle = E_0(|0\rangle + \sqrt{2}|2\rangle)$$

So
$$\begin{pmatrix} \langle 0|H_1|0\rangle & \langle 0|H_1|1\rangle \\ \langle 1|H_1|0\rangle & \langle 1|H_1|1\rangle \end{pmatrix} = E_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Eigenstates of this are $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ with eigenvalues $\pm E_0$, so the two lowest energies to first order in pert. theory are $\pm\lambda E_0$.

b) For $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle$, we have:

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}} \cdot e^{-iE_1 t/\hbar}|1\rangle + \frac{1}{\sqrt{2}} e^{-i\frac{E_2}{\hbar}t}|2\rangle$$

$$= \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}} e^{-i\frac{2E_0}{\hbar}t}|2\rangle$$

$$\langle\Psi(t)|x|\Psi(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{1}{\sqrt{2}}\langle 1| + \frac{1}{\sqrt{2}} e^{\frac{2iE_0 t}{\hbar}}\langle 2|\right)(a+a^{+})\left(\frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}} e^{-\frac{2iE_0 t}{\hbar}}|2\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{1}{\sqrt{2}}\langle 1| + \frac{1}{\sqrt{2}} e^{\frac{2iE_0 t}{\hbar}}\langle 2|\right)\left(\frac{1}{\sqrt{2}}|0\rangle + |2\rangle + e^{-\frac{2iE_0 t}{\hbar}}|1\rangle + \sqrt{\frac{3}{2}}\, e^{-\frac{2iE_0 t}{\hbar}}|3\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{1}{\sqrt{2}} e^{-\frac{2iE_0 t}{\hbar}} + \frac{1}{\sqrt{2}} e^{\frac{2iE_0 t}{\hbar}}\right)$$

$$= \sqrt{\frac{\hbar}{m\omega}} \cos\left(\frac{2E_0 t}{\hbar}\right)$$

Long Answer Question 3 (3 points)

Suppose that a unitary transformation $\hat{T}$ is a symmetry of a quantum system with Hamiltonian $\hat{H}$. Show that if $|E\rangle$ is an energy eigenstate with energy E then $\hat{T}|E\rangle$ is also an energy eigenstate with energy E.

Since $\hat{T}$ is a symmetry, we have that $[\hat{T}, \hat{H}] = 0 \Rightarrow \hat{T}\hat{H} = \hat{H}\hat{T}$

Since $|E\rangle$ is an energy eigenstate, we have $\hat{H}|E\rangle = E|E\rangle$.

Now: $\hat{H}\left[\hat{T}|E\rangle\right] = \hat{T}\hat{H}|E\rangle$ (since $\hat{T}$ and $\hat{H}$ commute)

$$= \hat{T}(E|E\rangle)$$

$$= E\left[\hat{T}|E\rangle\right]$$

Thus $\hat{T}|E\rangle$ is an eigenstate of $\hat{H}$ with energy E.