Name & Student number:

Physics 402 Midterm, Feb 28, 2019

Multiple Choice Questions: Please write your answers in the spaces on page 2

1) A spin half particle is in an eigenstate of S_x with eigenvalue $\hbar/2$. If we measure the z component of spin for this particle,

a) we will definitely find $S_z = 0$.

b) we might find any value between $S_z = -\hbar/2$ and $S_z = \hbar/2$

c) We will find either $S_z = -\hbar/2$ or $S_z = \hbar/2$ ← we always find one of the eigenvalues

2) A quantum system is in an eigenstate of an observable $\mathcal{O}$ with eigenvalue λ at some time $t = 0$. If we measure this observable at a later time $t = T$,

a) We will always find λ. b) We will not always find λ.

will always find λ if $[\mathcal{O}, H] = 0$ but not generally

3) For a time-independent Hamiltonian, the expectation value of energy stays constant in time

$[H, H] = 0$

$\Rightarrow$ energy conserved

a) for all states.

b) for all energy eigenstates but not for general states.

c) only for energy eigenstates with energy 0.

d) for no states.

4) Suppose that $|\Psi_1\rangle$ is some state of a particular molecule. If J_z is the operator associated with the z component of angular momentum, which of the following states states is the result of rotating the state $|\Psi_1\rangle$ about the z axis by angle $\pi/2$?

$(\mathbb{1} - i\varepsilon J_z)$ is infinitesimal rotation

a) $\frac{\pi}{2\hbar} J_z |\Psi_1\rangle$ b) $|\Psi_1\rangle - i\frac{\pi}{2\hbar} J_z |\Psi_1\rangle$ c) $e^{-i\frac{\pi}{2\hbar} J_z} |\Psi_1\rangle$ d) $|\Psi_1\rangle + e^{-i\frac{\pi}{2\hbar} J_z} |\Psi_1\rangle$

← repeat many times for $\varepsilon \to 0$ to get this

5) For a particle in the ground state of an infinite square well potential located at $[-L, L]$, the first order shift in the energy under a perturbation $\delta V(x)$ to the potential will vanish

a) if $\delta V(x)$ is an even function.

b) if $\delta V(x)$ is an odd function.

c) only if $\delta V(x)$ is a constant function.

d) for no nonzero function.

$$\delta E = \int_{-L}^{L} \underbrace{|\psi(x)|^2}_{\text{even}} \underbrace{\delta V(x)}_{\text{odd}} dx = 0$$

6) Which of the following expressions is the wavefunction for the ground state of a harmonic oscillator?

a) $\langle 0|\hat{x}|0\rangle$ b) $\hat{x}|0\rangle$ c) $\langle x|0\rangle$ d) $\langle 0|a|0\rangle$ e) $\langle 0|a^\dagger|0\rangle$

7) If Hermitian operators $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ commute with each other, one consequence is that

a) The observables $\mathcal{A}$ and $\mathcal{B}$ are conserved.

b) There is some basis of the Hilbert space for which the basis elements are eigenstates of both $\mathcal{A}$ and $\mathcal{B}$

c) It's possible to find a basis of energy eigenstates that have definite values for $\mathcal{A}$ and $\mathcal{B}$.

d) All states have definite values for $\mathcal{A}$ and $\mathcal{B}$.

8) If $[\hat{O}, H] = 0$, where H is the Hamiltonian and $\hat{O}$ is some Hermitian operator, which of the following is generally true?

a) All states have a definite value for both energy and the observable O.

b) If $|E\rangle$ is an energy eigenstate, then all other energy eigenstates are obtained by acting with $\hat{O}$ on $|E\rangle$ multiple times.

c) If we measure O at some time and repeat the measurement at a later time, we will always find the same result.

measure $O \longrightarrow$ gives state w. definite value of O

$[O, H] = 0 \Rightarrow$ probabilities for measurements of O unchanging in time

9) For a harmonic oscillator, the first order shift in the state $|3\rangle$ upon adding a perturbation λp^2 is:

a) $\lambda\langle 3|p^2|3\rangle$

b) $\lambda(|1\rangle\langle 1|H_1|3\rangle + |5\rangle\langle 5|H_1|3\rangle)$

c) $\lambda(|1\rangle\langle 1|H_1|3\rangle + |3\rangle\langle 3|H_1|3\rangle + |5\rangle\langle 5|H_1|3\rangle)$

d) $\frac{\lambda}{2\hbar\omega}(|1\rangle\langle 1|H_1|3\rangle - |5\rangle\langle 5|H_1|3\rangle)$

e) $\frac{\lambda}{2\hbar\omega}(|1\rangle|\langle 1|H_1|3\rangle|^2 - |5\rangle|\langle 5|H_1|3\rangle|^2)$

10) True or false: for any quantum mechanical system, if a state $|\Psi_1\rangle$ at $t = 0$ evolves to state $|\hat{\Psi}_1\rangle$ at time $t = 1s$ and a state $|\Psi_2\rangle$ at $t = 0$ evolves to state $|\hat{\Psi}_2\rangle$ at time $t = 1s$, then a state $c_1|\Psi_1\rangle + c_2|\Psi_2\rangle$ at $t = 0$ would necessarily evolve to state $c_1|\hat{\Psi}_1\rangle + c_2|\hat{\Psi}_2\rangle$ at time $t = 1s$.

a) True

b) False

Write your multiple choice answers here:

1	2	3	4	5
6	7	8	9	10

Long Answer Question 1

a) For a quantum system of a particle in one dimension, how is the physical transformation of spatial translation related to the physical observable momentum?

Translation by a distance a acts as a unitary operation $\hat{\tau}(a)$ on the Hilbert space. For small a, we can write

$$\hat{\tau}(a) = \mathbb{1} - i\frac{a}{\hbar}\hat{P} + \ldots$$

The operator $\hat{P}$ is Hermitian and corresponds to the physical observable momentum.

b) Explain how this relation can be used to derive the wavefunction for the state $\hat{P}|\Psi\rangle$ given the wavefunction $\psi(x)$ for the state $|\Psi\rangle$.

Since $\hat{\tau}(a)$ gives a translation, the wavefunction for $\hat{\tau}(a)|\Psi\rangle$ is $\psi(x-a)$. So

$$\langle x|\hat{\tau}(a)|\Psi\rangle = \psi(x-a)$$

Expanding both sides for small a, we get

$$\langle x|\mathbb{1} - \frac{i}{\hbar}a\hat{P} + \ldots|\Psi\rangle = \psi(x) - a\psi'(x) + \ldots$$

So $\langle x|\hat{P}|\Psi\rangle = \frac{\hbar}{i}\psi'(x)$.

This is the wavefunction for the state $\hat{P}|\Psi\rangle$.

Long Answer Question 2

Three spin 1/2 particles sit at fixed positions in a linear configuration. The interactions between the particles give rise to a Hamiltonian

$$H = A(S_1^z S_2^z + S_2^z S_3^z) \ ,$$

where $A > 0$ and for example, S_z^1 is the z-component of the spin of the first particle. (Note: in tensor product notation, $S_1^z S_2^z = S_1^z \otimes S_2^z \otimes \mathbb{1}$).

a) Write down a basis of energy eigenstates for this system and the corresponding energy eigenvalues.

The states with a definite value of S_z for each of the three spins will be energy eigenstates. For example:

$$H|\uparrow\uparrow\uparrow\rangle = A\cdot\left(\frac{\hbar}{2}\cdot\frac{\hbar}{2} + \frac{\hbar}{2}\cdot\frac{\hbar}{2}\right) = \frac{1}{2}A\hbar^2$$

The remaining energy eigenstates are:

$|\downarrow\downarrow\downarrow\rangle$: $E = \frac{1}{2}A\hbar^2$

$|\uparrow\uparrow\downarrow\rangle$: $E = 0$

$|\downarrow\downarrow\uparrow\rangle$: $E = 0$

$|\uparrow\downarrow\downarrow\rangle$: $E = 0$

$|\downarrow\uparrow\uparrow\rangle$: $E = 0$

$|\uparrow\downarrow\uparrow\rangle$: $E = -\frac{1}{2}A\hbar^2$

$|\downarrow\uparrow\downarrow\rangle$: $E = -\frac{1}{2}A\hbar^2$

b) A new interaction perturbs the Hamiltonian by

$$H \to H + b(S_1^x S_2^x S_3^x) \; . \tag{1}$$

$\xleftarrow{} H_1$

where b is a parameter. If $E_0(b)$ is the ground state energy as a function of b, determine $E_0(b)$ up to first order in the parameter b.

c) For what range of the parameter b do you expect that your first order result will be reliable?

b) We have that $E_0(b=0) = -\frac{1}{2}A\hbar^2$, and this level is degenerate, with states $|\uparrow\downarrow\uparrow\rangle$ and $|\downarrow\uparrow\downarrow\rangle$ having the same energy. To find the first order shifts for these states, we need degenerate perturbation theory. Using that

$$S_x|\uparrow\rangle = \frac{\hbar}{2}|\downarrow\rangle \text{ and } S_x|\downarrow\rangle = \frac{\hbar}{2}|\uparrow\rangle,$$

we get: $H_1|\uparrow\downarrow\uparrow\rangle = b\left(\frac{\hbar}{2}\right)^3|\downarrow\uparrow\downarrow\rangle \quad H_1|\downarrow\uparrow\downarrow\rangle = b\left(\frac{\hbar}{2}\right)^3|\uparrow\downarrow\uparrow\rangle$

$$\begin{pmatrix} \langle\uparrow\downarrow\uparrow|H_1|\uparrow\downarrow\uparrow\rangle & \langle\downarrow\uparrow\downarrow|H_1|\uparrow\downarrow\uparrow\rangle \\ \langle\uparrow\downarrow\uparrow|H_1|\downarrow\uparrow\downarrow\rangle & \langle\downarrow\uparrow\downarrow|H_1|\downarrow\uparrow\downarrow\rangle \end{pmatrix} = b\left(\frac{\hbar}{2}\right)^3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This has eigenvalues $\pm b\left(\frac{\hbar}{2}\right)^3$, and these are the 1st order energy shifts for the states with energy $-\frac{1}{2}A\hbar^2$. The ground state energy is then

$$E_0(b) = -\frac{1}{2}A\hbar^2 - b\left(\frac{\hbar}{2}\right)^3 + \ldots \quad (\text{assuming } b>0)$$

c) We expect this to be reliable when the first order correction is small compared to the unperturbed energy, so:

$$b\left(\frac{\hbar}{2}\right)^3 \ll \frac{1}{2}A\hbar^2$$

or $\boxed{b \ll \frac{4A}{\hbar}}$

Long Answer Question 3

A 1D harmonic oscillator is in the state

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

at time $t = 0$. What is the expectation value of the particle's position at time T?

b) If we measure the particle's position at time T, roughly how close do we expect to find it to this expectation value (on average)?

a) The states $|0\rangle$ and $|1\rangle$ are energy eigenstates with energies $\frac{\hbar\omega}{2}$ and $\frac{3\hbar\omega}{2}$. Under time evolution, they evolve as:

$$|0\rangle \to e^{-\frac{iHt}{\hbar}}|0\rangle = e^{-\frac{i\omega t}{2}}|0\rangle$$

$$|1\rangle \to e^{-\frac{iHt}{\hbar}}|1\rangle = e^{-\frac{3}{2}i\omega t}|1\rangle$$

By linearity of the Schrödinger equation, the state $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$ evolves to

$$|\Psi(T)\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{i\omega T}{2}}|0\rangle + e^{-\frac{3i\omega T}{2}}|1\rangle\right)$$

at time T. The expectation value of position at this time is: $\langle\Psi(T)|\hat{x}|\Psi(T)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle\Psi(T)|a+a^\dagger|\Psi(T)\rangle.$

The non-zero terms in this expression are

$$\sqrt{\frac{\hbar}{2m\omega}}\left(\frac{1}{2}e^{i\omega T/2}\langle 0|a|1\rangle\cdot e^{-\frac{3i\omega T}{2}} + \frac{1}{2}e^{\frac{3i\omega T}{2}}\langle 1|a^\dagger|0\rangle e^{-\frac{i\omega T}{2}}\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{1}{2}\left(e^{-i\omega T} + e^{i\omega T}\right)\right) = \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega T) \leftarrow \text{call this } \bar{x}$$

b) We can estimate this by taking $\langle(x-\bar{x})^2\rangle$ and then taking the square root (this is the uncertainty in x). Have:

$$\langle(x-\bar{x})^2\rangle = \langle x^2 - 2x\bar{x} + \bar{x}^2\rangle = \langle x^2\rangle - \bar{x}^2$$

Now, $\langle x^2\rangle = \frac{\hbar}{2m\omega}\langle(a+a^\dagger)^2\rangle$. Also: $(a+a^\dagger)|\Psi(T)\rangle$

$$= \frac{1}{\sqrt{2}}\left(e^{-\frac{i\omega T}{2}}|1\rangle + e^{-\frac{3i\omega T}{2}}|0\rangle + \sqrt{2}e^{-\frac{3i\omega T}{2}}|2\rangle\right)$$

So $\langle x^2\rangle = \frac{\hbar}{2m\omega}\cdot 2$. Thus $\sqrt{\langle(x-\bar{x})^2\rangle} = \sqrt{\frac{\hbar}{2m\omega}[2-\cos^2\omega T]} = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{1+\sin^2\omega T}$