Name: SOLUTIONS

Student Number:

Physics 402 Midterm, March 1, 2018

Multiple Choice Questions: (one point each)
Please write your answers in the spaces on page 2

Assume that all states are properly normalized unless otherwise specified.

1) A quantum harmonic oscillator with mass m and frequency ω is in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle) .$$

Have $E_n = \hbar\omega(n+\frac{1}{2})$
- always find one of the energy eigenvalues if we measure.

If we measure the energy, what values might we obtain?
a) Any value of energy is possible.
b) Any value of energy between $\hbar\omega/2$ and $5\hbar\omega/2$ is possible, but $3\hbar\omega/2$ is most likely.
c) We will find either $\hbar\omega/2$, $3\hbar\omega/2$, or $5\hbar\omega/2$
d) The result will be $3\hbar\omega/2$.

2) For an observable $\mathcal{O}$ in a quantum system, an eigenstate of $\mathcal{O}$ is
a) A value of $\mathcal{O}$ that we might obtain in a measurement.
b) A state with a definite value for the quantity $\mathcal{O}$.
c) A state for which the value of $\mathcal{O}$ is 0.
d) A state for which the value of $\mathcal{O}$ does not change with time.

3) For a state $|\psi\rangle$ and a Hermitian operator $\hat{B}$ associated with an observable B, suppose that $\langle\psi|\hat{B}|\psi\rangle = 4$. Then we can say that → this is expectation value = avg over a large number of measurements
a) The observable B has a definite value of 4 in the state $|\psi\rangle$.
b) The observable B does not necessarily have a definite value of B before measuring, but we will find the value 4 if we measure B.
c) The observable B does not necessarily have a definite value of B before measuring, but the average result for a large number of measurements of B on states identical to $|\psi\rangle$ will be 4.
d) The observable B does not necessarily have a definite value of B; the quantity $\langle\psi|\hat{B}|\psi\rangle$ does not have any direct connection to the results of measurements of B.

4) If Hermitian operators $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ commute with each other, one consequence is that
a) The observables $\mathcal{A}$ and $\mathcal{B}$ are conserved.
b) The observables $\mathcal{A}$ and $\mathcal{B}$ are equal to each other for all states.
c) It is possible for a state to have a definite value of both $\mathcal{A}$ and $\mathcal{B}$, but this is not necessarily true for every state.
d) All states have definite values for $\mathcal{A}$ and $\mathcal{B}$.

formula sheet on back

5) Given a state $|\Psi\rangle$ in a quantum mechanical system with energy operator $\hat{\mathcal{H}}$, the change in the state after an infinitesimal change δt in time is

a) $\frac{\delta t}{i\hbar}\hat{\mathcal{H}}|\Psi\rangle$

b) $\langle\Psi|\hat{\mathcal{H}}|\Psi\rangle\delta t$

c) $e^{i\hat{\mathcal{H}}/\hbar\delta t}|\Psi\rangle$

d) $\hat{\mathcal{H}}\delta t$

Schrödinger: $i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle$

So: $|\psi(t+\delta t)\rangle - |\psi(t)\rangle = \delta t\cdot\frac{\hat{H}}{i\hbar}|\psi\rangle$

6) For some Hermitian operator $\hat{O}$ that commutes with the Hamiltonian, we can say that

a) The quantity $\hat{O}|\Psi\rangle$ is proportional to the infinitesimal change in the state $|\Psi\rangle$ under time evolution.

$\Rightarrow$ O conserved.

b) All states will have definite values for $\hat{O}$.

c) The ground state of the Hamiltonian will also have $\mathcal{O} = 0$.

d) A state with some expectation value for $\hat{O}$ will continue to have that value.

7) For a quantum harmonic oscillator, the spacing in energy between successive energy eigenstates

a) is the same at all energies.

b) increases with increasing energy.

c) decreases with increasing energy.

d) None of the above: the allowed energies are continuous for this system.

$E_n = \hbar\omega(n+\frac{1}{2})$

8) If $|\Psi(\lambda)\rangle$ is an energy eigenstate for Hamiltonian $H_0 + \lambda H_1$ with energy E_0 for $\lambda = 0$, the energy of $|\Psi(\lambda)\rangle$, expressed as a power series in λ takes the form

a) $E_0 + \lambda\langle\Psi(0)|H_1|\Psi(0)\rangle + \ldots$

b) $E_0 + \lambda|\langle\Psi(0)|H_1|\Psi(0)\rangle|^2 + \ldots$

c) $E_0 + \frac{\lambda\langle\Psi(0)|H_1|\Psi(0)\rangle}{E_0-E_1} + \ldots$

d) $E_0 + \frac{\lambda|\langle\Psi(0)|H_1|\Psi(0)\rangle|^2}{E_0-E_1} + \ldots$

1st order perturbation theory

9) If we measure the z component of angular momentum for a spin 1/2 particle, how many possible measurement outcomes are there?

a) 0   b) 1   c) 2   d) 3   e) ∞

$J_z = \frac{\hbar}{2}, -\frac{\hbar}{2}$

10) In order to give a position-space description of the state of a quantum system with two particles in one dimension, we can use

a) two wavefunctions $\psi_1(x)$ and $\psi_2(x)$, one for each particle.

b) a single wavefunction $\psi(x_1, x_2)$ depending on two position variables.

c) two wavefunctions, $\psi_1(x_1, x_2)$ and $\psi_2(x_1, x_2)$ each depending on two variables.

d) Either a) or b).

Basis: $|x_1, x_2\rangle$ General state: $\int dx_1 dx_2\,\psi(x_1,x_2)\,|x_1,x_2\rangle$

Please fill in:

Answers

1	2	3	4	5	6	7	8	9	10
c	b	c	c	a	d	a	a	c	b

Long Answer Question 1 (5 points)

"Energy is conserved in a quantum system with time-translation invariance."

a) Explain what this statement means mathematically and to which systems it applies.

→ For any state $|\Psi(t)\rangle$ the expectation value of energy $\langle\Psi(t)|\hat{H}|\Psi(t)\rangle$ is independent of time. Equivalently, for any state $|\Psi(t)\rangle$, the probability for each measurement outcome in a measurement of energy is independent of time.

This applies when the Hamiltonian is independent of time, or equivalently when the time evolution operator doesn't depend on the initial time, but only on the elapsed time.

b) Explain how this statement can be derived starting from the basic assumption that time evolution in a quantum mechanical system corresponds to a unitary transformation on the state. (*If you are stuck, at least try to give some kind of argument for why this statement is true, or how you could demonstrate it mathematically.*)

Time evolution is a unitary operation, so

$$|\Psi(t)\rangle = \mathcal{U}(t_0, \Delta t)|\Psi(0)\rangle$$

For a time-translation invariant system, $\mathcal{U}(t_0, \Delta t) = \mathcal{U}(\Delta t)$. Infinitesimal time translations are unitary operators close to the identity. These take the form $\mathcal{U}(\delta t) = \mathbb{1} - \frac{i}{\hbar}\delta t\, \hat{H}$ for some Hermitian operator $\hat{H}$. The change in the state under an infinitesimal time evolution is then:

$$\delta|\Psi\rangle = -\frac{i}{\hbar}\delta t\,\hat{H}\,|\Psi\rangle$$

Since $\hat{H}$ is Hermitian, it corresponds to a physical observable - we call this energy. The change in the expectation value of energy under an infinitesimal time translation is

$$\delta\langle\Psi|\hat{H}|\Psi\rangle = \left(\frac{i}{\hbar}\delta t\,\langle\Psi|\hat{H}\right)\hat{H}|\Psi\rangle + \langle\Psi|\hat{H}\left(-\frac{i}{\hbar}\delta t\,\hat{H}|\Psi\rangle\right)$$

$$= 0$$

So energy is conserved.

(Alternatively: $\langle\Psi(t)|\hat{H}|\Psi(t)\rangle = \langle\Psi(0)|e^{i\hat{H}t/\hbar}\,\hat{H}e^{-i\hat{H}t/\hbar}|\Psi(0)\rangle = \langle\Psi(0)|\hat{H}|\Psi(0)\rangle$ since $[\hat{H}, e^{-i\hat{H}t/\hbar}] = 0$)

Long Answer Question 2 (6 points)

A quantum system consists of a spin 1 particle with a Hamiltonian that can be adjusted by changing a magnetic field in the system. The Hamiltonian can be written as

λ^2 terms

$$H(\lambda) = E_0(e^{\lambda}J_z + e^{2\lambda}J_z^2 + 2\lambda J_x)$$
$$= E_0(J_z + J_z^2) + \lambda E_0(J_z + 2J_z^2 + 2J_x) + \ldots$$

where λ is the parameter that we can control (for this question, assume we are using units where $\hbar = 1$). This system has energy eigenstates $|\Psi_i(\lambda)\rangle$ ($i = 1, 2, 3$) with energies $E_i(\lambda)$ which we would like to estimate for small λ. Determine the states $|\Psi_i(\lambda)\rangle$ in the limit where $\lambda \to 0$ and determine the corresponding energies $E_i(\lambda)$ up to first order in λ. Express the states using the J_z basis. (*Hint: a good start would be to find energy eigenvalues and eigenstates for* $\lambda = 0$).

For $\lambda=0$, we have: $H(\lambda)\big|_{\lambda=0} = E_0(J_z + J_z^2)$. The J_z eigenstates $|M\rangle$ will also be eigenstates of $H(0)$. We have:

$$H(0)|M\rangle = E_0(J_z + J_z^2)|M\rangle$$
$$= E_0(M + M^2)|M\rangle$$

So: $|1\rangle$ has energy $2E_0$, $|0\rangle$ and $|-1\rangle$ have energy 0.

To find the states and energies for small non-zero λ, we use perturbation theory. To order λ, the correction to the Hamiltonian is: λH_1 where:

$$H_1 = \frac{d}{d\lambda}H(\lambda)\Big|_{\lambda=0} = E_0(J_z + 2J_z^2 + 2J_x)$$
$$= E_0(J_z + 2J_z^2 + J_+ + J_-)$$

For the state $|1\rangle$, the first order shift is

$$\lambda\langle 1|H_1|1\rangle = \lambda E_0\langle 1|J_z + 2J_z^2|1\rangle = 3\lambda E_0$$

so the energy to order λ is $2E_0 + 3\lambda E_0$.

For $|0\rangle$ and $|-1\rangle$, we need degenerate perturbation theory. We have:

$$\begin{pmatrix} \langle 0|H_1|0\rangle & \langle 0|H_1|-1\rangle \\ \langle -1|H_1|0\rangle & \langle -1|H_1|-1\rangle \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$$

using: $J_+|-1\rangle = \sqrt{J\cdot(J+1) - M(M+1)}\,|0\rangle = \sqrt{2}\,|0\rangle$, etc...

This has eigenvalues given by the characteristic equation:

$$0 = \det\begin{pmatrix} \lambda & -\sqrt{2} \\ -\sqrt{2} & \lambda-1 \end{pmatrix} = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$. The corresponding eigenvectors are: $v_1 = \frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{2}\\2\end{pmatrix}$ and $v_2 = \frac{1}{\sqrt{3}}\begin{pmatrix}\sqrt{2}\\-1\end{pmatrix}$.

So we can say that the state

$$\frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|-1\rangle \text{ gets shift } \lambda \cdot 2E_0$$

and

$$\sqrt{\frac{2}{3}}|0\rangle - \sqrt{\frac{1}{3}}|-1\rangle \text{ gets shift } -\lambda \cdot E_0.$$

Long Answer Question 3 (5 points)

Consider a particle in a two-dimensional harmonic oscillator potential. The Hamiltonian is

$$H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2 y^2 \tag{1}$$

and we have energy eigenstates $|n_x\, n_y\rangle$ with energy $\hbar\omega(n_x + n_y + 1)$.

a) A basis of states with energy $3\hbar\omega$ is $\{|20\rangle, |11\rangle, |02\rangle\}$. If we modify the system so that $H_1 = \frac{1}{2}\alpha(xp_y - yp_x)$ is added to the Hamiltonian, three different linear combinations of the form

↖ this was a typo.

$$A|20\rangle + B|11\rangle + C|02\rangle \tag{2}$$

are *exact* energy eigenstates of $H_0 + H_1$, with energies $\{3\hbar\omega + \alpha\hbar, 3\hbar\omega, 3\hbar\omega - \alpha\hbar\}$. Determine which linear combinations give each of these energies.

b)If the state of system at time $t = 0$ is $|11\rangle$ and we measure n_y at time T, what is the probability that we will find $n_y = 2$ (assume the Hamiltonian is $H_0 + H_1$)?

We want to find eigenstates of $H_0 + H_1$ of the form

$$A|20\rangle + B|11\rangle + C|02\rangle$$

First, we know that H_0 acting on any of these gives $3\hbar\omega$. To understand how H_1 acts, we use:

$$X = \sqrt{\frac{\hbar}{2m\omega}}\,(a_x + a_x^\dagger) \qquad P_x = -i\sqrt{\frac{\hbar m\omega}{2}}\,(a_x - a_x^\dagger)$$

$$Y = \sqrt{\frac{\hbar}{2m\omega}}\,(a_y + a_y^\dagger) \qquad P_y = -i\sqrt{\frac{\hbar m\omega}{2}}\,(a_y - a_y^\dagger)$$

Then: $$H_1 = \frac{1}{2}\alpha\left(-i\frac{\hbar}{2}\right)\left[(a_x + a_x^\dagger)(a_y - a_y^\dagger) - (a_y + a_y^\dagger)(a_x - a_x^\dagger)\right]$$

$$= \frac{1}{2}\alpha\left(-i\frac{\hbar}{2}\right)\left[a_x^\dagger a_y - a_x a_y^\dagger + a_y a_x^\dagger - a_y^\dagger a_x\right]$$

$$= \frac{1}{2} i\hbar\alpha\,(a_y^\dagger a_x - a_x^\dagger a_y)$$

Then: $$H_1|20\rangle = \frac{1}{2} i\hbar\alpha\sqrt{2}\,|1\,1\rangle$$

$$H_1|11\rangle = \frac{1}{2} i\hbar\alpha\sqrt{2}\,(-|2\,0\rangle + |0\,2\rangle)$$

$$H_1|02\rangle = -\frac{i}{2}\hbar\alpha\sqrt{2}\,|1\,1\rangle$$

using $a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

According to the question, there are eigenstates of the form

$$A|20\rangle + B|11\rangle + C|02\rangle$$

with eigenvalues $\alpha\hbar$, 0, and $-\alpha\hbar$.

For $\lambda = 0$, we have:

$$H_1(A|20\rangle + B|11\rangle + C|02\rangle) = 0$$

$$\Rightarrow \frac{1}{2} i\hbar\alpha\left[(A\sqrt{2} - C\sqrt{2})|11\rangle + B\sqrt{2}|02\rangle - B\sqrt{2}|20\rangle\right] = 0$$

$$\Rightarrow B = 0 \text{ and } A = C$$

So state $\frac{1}{\sqrt{2}}(|20\rangle - |02\rangle)$ has energy $3\hbar\omega$

For $\lambda = \alpha\hbar$, we have:

$$\frac{1}{2} i\hbar\alpha\left[(A\sqrt{2} - C\sqrt{2})|11\rangle + B\sqrt{2}|02\rangle - B\sqrt{2}|20\rangle\right] = \alpha\hbar\left[A|20\rangle + B|11\rangle + C|02\rangle\right]$$

so: $\frac{i\sqrt{2}}{2}(A - C) = B \quad iB\frac{\sqrt{2}}{2} = A$ and $-iB\frac{\sqrt{2}}{2} = C \Rightarrow (A, B, C) = B\left(i\frac{\sqrt{2}}{2}, 1, -\frac{i\sqrt{2}}{2}\right)$

So state

$$|\Phi_+\rangle \equiv \frac{1}{\sqrt{2}}|20\rangle - \frac{i}{2}|11\rangle - \frac{1}{\sqrt{2}}|02\rangle \text{ has energy } 3\hbar\omega + \alpha\hbar$$

For $\lambda = \alpha\hbar$, we similarly get:

$$\frac{i\sqrt{2}}{2}(A - C) = -B \quad iB\frac{\sqrt{2}}{2} = -A \quad -iB\frac{\sqrt{2}}{2} = -C \Rightarrow (A, B, C) = B\left(-\frac{i\sqrt{2}}{2}, 1, \frac{i\sqrt{2}}{2}\right)$$

So state $|\Phi_-\rangle \equiv \frac{1}{\sqrt{2}}|20\rangle + \frac{i}{2}|11\rangle - \frac{1}{\sqrt{2}}|02\rangle$ has energy $3\hbar\omega + \alpha\hbar$.

b) Our initial state is $|11\rangle = i(|\Phi_+\rangle - |\Phi_-\rangle)$. At time T, we have

write in terms of energy eigenstates

$$|\Phi(t)\rangle = e^{-i\hat{H}T/\hbar}(i|\Phi_+\rangle - i|\Phi_-\rangle)$$

$\hat{H}$ gets replaced by eigenvalue.

$$= i e^{-i\cdot 3\hbar\omega T/\hbar}\left(e^{-i\alpha T}|\Phi_+\rangle - e^{i\alpha T}|\Phi_-\rangle\right)$$

$$= i e^{-i 3\hbar\omega T/\hbar}\left(-\frac{1}{\sqrt{2}}\right)\left(e^{-i\alpha T} - e^{i\alpha T}\right)|02\rangle + \text{other terms w. } n_y \neq 2.$$

$$\text{So: } P_{n_y=2}(T) = \left|\frac{1}{\sqrt{2}}\left(e^{i\alpha T} - e^{-i\alpha T}\right)\right|^2 = 2\sin^2(\alpha T)$$

$|\text{coeff of } |02\rangle \text{ state}|^2$