Formula sheet at the back

SOLUTIONS

Name: Mark
Student Number:

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1) If $|A\rangle = \frac{i}{2}|\uparrow\rangle - \frac{\sqrt{3}}{2}|\downarrow\rangle$, what is $\langle A|\uparrow\rangle$?

a) $i/2$

b) $-i/2$

c) $-\sqrt{3}/2$

d) $\sqrt{3}/2$

We have $\langle A| = -\frac{i}{2}\langle\uparrow| - \frac{\sqrt{3}}{2}\langle\downarrow|$

so $\langle A|\uparrow\rangle = -\frac{i}{2}$

2) Which of the following represents the average value of energy E that we would obtain if we made measurements of energy on many identical states $|\Psi\rangle$?

a) $\hat{H}|\Psi\rangle$

b) $\langle\Psi|\hat{H}|\Psi\rangle$

c) $\langle E|\Psi\rangle$

d) $|\langle E|\Psi\rangle|^2$

e) ΔE

3) If $\hat{O}_1$ and $\hat{O}_2$ are both time-independent Hermitian operators, which of the following is a consequence of $[\hat{O}_1, \hat{O}_2] = 0$?

a) The expectation value of the physical observables associated with $\hat{O}_1$ and $\hat{O}_2$ are unchanging in time.

b) We can find a basis of states which have definite values for the quantities $\mathcal{O}_1$ and $\mathcal{O}_2$.

c) The operation $(\mathbb{1} - i\epsilon\mathcal{O}_1)$ is a symmetry of the theory.

d) All of the above.

4) The state $|3\rangle$ of the quantum harmonic oscillator with energy $7/2\hbar\omega$, is an eigenstate of which operator?

Have $a^\dagger a|3\rangle = a^\dagger(\sqrt{3}|2\rangle) = 3|3\rangle$

a) a b) $a^\dagger$ c) $a^\dagger a$ d) $a + a^\dagger$

5) A quantum system with a time-independent Hamiltonian is in some energy eigenstate $|E\rangle$ at time $t = 0$, which of the following is NOT true about this state

a) The energy expectation value will be independent of time.

b) All physical observables will be independent of time.

c) If we measure the energy at any later time, we will always find E.

d) The position space probability density for the state will oscillate periodically with a specific frequency. $\rightarrow$ wavefn. oscillates as $e^{-iEt/\hbar}\psi(x)$ but $|\psi|^2$ is constant.

Multiple Choice Answers:

#1	#2	#3	#4	#5	#6	#7	#8	#9	#10
b	b	b	c	d	c	c	b	c	d

6) For the linear combination $\frac{1}{\sqrt{14}}(|E_1\rangle + 2|E_2\rangle + 3|E_3\rangle)$ of energy eigenstates, suppose we measure the energy. The probability that we will find E_2 is

a) 1/14

$P_{E_2} = |\langle E_2|\Psi\rangle|^2 = \left(\frac{2}{\sqrt{14}}\right)^2 = \frac{2}{7}.$

b) 1/7

c) 2/7

d) 4/7

e) 0

7) If $\hat{\mathcal{P}}$ is the momentum operator, which of the states below represent the state $|\Psi\rangle$ of a particle in one dimension translated to the right by a non-infinitesimal amount a?

a) $a\hat{\mathcal{P}}|\Psi\rangle$

b) $(1 - ia\hat{\mathcal{P}}/\hbar)|\Psi\rangle$

c) $e^{-ia\hat{\mathcal{P}}/\hbar}|\Psi\rangle$

d) $\langle a|\hat{\mathcal{P}}|\Psi\rangle$

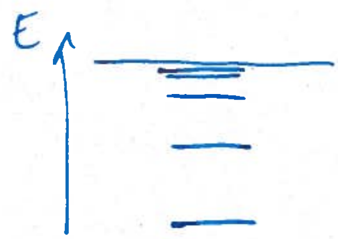
8) For the states of an electron bound in a hydrogen atom, we can say that the possible energies are

a) equally spaced

b) farther apart for lower energies

c) farther apart for higher energies

d) all continuous



9) If we add a perturbation λx^3 to a harmonic oscillator the first *nonzero* correction to the energy of the state $|0\rangle$ is

a) $\lambda\langle 1|x^3|1\rangle$

b) $\lambda\langle 0|x^3|0\rangle$

c) $\sum_{n=1}^{\infty} \frac{|\lambda\langle n|x^3|0\rangle|^2}{E_0 - E_n}$

d) $\lambda \sum_{n=1}^{\infty} |n\rangle \frac{\langle n|x^3|0\rangle}{E_0 - E_n}$

10) Which of the following defines is the wavefunction $\psi(x)$ of a state $|\Psi\rangle$?

a) $\langle\Psi|\Psi\rangle$

b) $\langle\Psi|\hat{x}|\Psi\rangle$

c) $\hat{x}|\Psi\rangle$

d) $\langle x|\Psi\rangle$

Long Answer Question 1

a) In quantum mechanics, what is meant by the statement that a physical observable $\mathcal{O}$ is conserved (give one definition)? What property of the associated operator guarantees this to be true?

We can say $\mathcal{O}$ is conserved if for all states $|\Psi(t)\rangle$ satisfying the Schrödinger equation, the expectation value of $\mathcal{O}$ is independent of time. OR we can say that the probability for measuring any particular eigen value of $\mathcal{O}$ is independent of time for all states $|\Psi(t)\rangle$.

This is guaranteed by $[\hat{O}, \hat{H}] = 0$

b) What is meant by a symmetry in quantum mechanics? How are symmetries represented mathematically in quantum mechanics?

A symmetry is a unitary transformation from states $\rightarrow$ states such that if $|\Psi(t)\rangle$ is a solution to the Schrödinger equation, then the transformed state $\hat{\tau}|\Psi(t)\rangle$ is also a solution

Long Answer Question 2 From special relativity, the exact formula for kinetic energy in terms of momentum is

$$E_K = \sqrt{m^2c^4 + p^2c^2} - mc^2 = \frac{p^2}{2m} - \frac{1}{8m^3c^2}p^4 + \dots \tag{1}$$

So we can take into account the leading effects of special relativity by adding $H_1 = -\frac{1}{8m^3c^2}p^4$ to the Hamiltonian for a system. For a particle of mass m in a 1D harmonic oscillator potential $m\omega^2/2$ in the first excited state $|1\rangle$, what is correction to the energy if we take into account this perturbation to first order?

We need to compute

$$\Delta E_1 = \langle 1| H_1 |1\rangle$$

$$= -\frac{1}{8m^3c^2} \langle 1| p^4 |1\rangle$$

We can write $p = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$ so $p^4 = \left(\frac{\hbar m\omega}{2}\right)^2 (a-a^\dagger)^4$

So $\Delta E_1 = -\dfrac{\hbar^2\omega^2}{32mc^2} \langle 1| (a-a^\dagger)^4 |1\rangle$

Using $a|n\rangle = \sqrt{n}\,|n-1\rangle$, $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$, we get:

$$(a-a^\dagger)^4|1\rangle = (a-a^\dagger)^3\left(-\sqrt{2}|2\rangle + |0\rangle\right)$$

$$= (a-a^\dagger)^2\left(-2|1\rangle + \sqrt{6}|3\rangle - |1\rangle\right) = (a-a^\dagger)^2\left(-3|1\rangle + \sqrt{6}|3\rangle\right)$$

$$= (a-a^\dagger)\left(-3|0\rangle + 3\sqrt{2}|2\rangle + 3\sqrt{2}|2\rangle + \sqrt{24}|4\rangle\right)$$

$$= 12|1\rangle + \sqrt{96}|3\rangle + 3|1\rangle - 6\sqrt{6}|3\rangle - \sqrt{120}|5\rangle$$

$$= 15|1\rangle + (\ )|3\rangle + (\ )|5\rangle$$

$$\langle 1|(a-a^\dagger)^4|1\rangle = 15$$

So the first order energy shift is $-\dfrac{15}{32}\dfrac{\hbar\omega^2}{mc^2}$

Long Answer Question 3 Consider a system of two spin half particles (at some fixed positions that we ignore), with Hamiltonian

$$H = -Y S_z^1 S_z^2 \,. \tag{2}$$

where Y is some positive constant and S_z^1, S_z^2, are the z-spin operators which act on the first and second spins respectively (without affecting the other spin).

a) Write a basis of energy eigenstates for this system and indicate the energy of each state.

We can use the basis of eigenstates of S_z^1 and S_z^2. We have:

$|\uparrow\uparrow\rangle$: energy $-Y\left(\frac{\hbar}{2}\right)^2 = -\frac{Y\hbar^2}{4}$

$|\downarrow\downarrow\rangle$: energy $-Y\left(-\frac{\hbar}{2}\right)^2 = -\frac{Y\hbar^2}{4}$

$|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$: energy $-Y\left(\frac{\hbar}{2}\right)\left(-\frac{\hbar}{2}\right) = Y\frac{\hbar^2}{4}$

b) Calculate the energy shift(s) for the ground state(s) when we add a perturbation $H_1 = \lambda S_y^1 S_y^2$.

The ground states are $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$. We need to use degenerate perturbation theory to calculate the 1st order energy shifts. First we calculate $\langle a_n|H_1|a_m\rangle$ for the degenerate states. We get:

$$\begin{pmatrix} \langle\uparrow\uparrow|H_1|\uparrow\uparrow\rangle & \langle\uparrow\uparrow|H_1|\downarrow\downarrow\rangle \\ \langle\downarrow\downarrow|H_1|\uparrow\uparrow\rangle & \langle\downarrow\downarrow|H_1|\downarrow\downarrow\rangle \end{pmatrix}$$

use $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ in $|\uparrow\rangle$ $|\downarrow\rangle$ basis of S_z eigenstates.

$$= \lambda \begin{pmatrix} \langle\uparrow|S_y|\uparrow\rangle\langle\uparrow|S_y|\uparrow\rangle & \langle\uparrow|S_y|\downarrow\rangle\langle\uparrow|S_y|\downarrow\rangle \\ \langle\downarrow|S_y|\uparrow\rangle\langle\downarrow|S_y|\uparrow\rangle & \langle\downarrow|S_y|\downarrow\rangle\langle\downarrow|S_y|\downarrow\rangle \end{pmatrix}$$

$$= \lambda\begin{pmatrix} 0 & -\hbar^2/4 \\ -\hbar^2/4 & 0 \end{pmatrix} = -\frac{\lambda\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This has eigenvalues $\pm\frac{\lambda\hbar}{4}$, so these are the first order energy shifts.

c) If we start in the state

$$|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\uparrow\rangle)\,, \tag{3}$$

what is the state after time T, and what is the probability that if we measure S_x, the first spin will have $S_x = \hbar/2$?

The time evolution operator is $e^{-i\hat{H}t/\hbar}$ so we have:

$$|\Psi(T)\rangle = e^{-i\hat{H}T/\hbar}\,|\Psi(0)\rangle$$

$$= e^{-i\hat{H}T/\hbar}\left(\frac{1}{\sqrt{2}}|\uparrow\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\hat{H}T/\hbar}|\uparrow\uparrow\rangle + e^{-i\hat{H}T/\hbar}|\downarrow\uparrow\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(e^{+iT\gamma\hbar/4}|\uparrow\uparrow\rangle + e^{-iT\gamma\hbar/4}|\downarrow\uparrow\rangle\right)$$

To find the probability that the first spin will be in state $S_x = \frac{\hbar}{2}$, regardless of what the second spin is doing we can say:

$$P_{S_x^1=\frac{\hbar}{2}} = P_{S_x^1=\frac{\hbar}{2},\,S_z^2=\frac{\hbar}{2}} + P_{S_x^1=\frac{\hbar}{2},\,S_z^2=-\frac{\hbar}{2}}$$

$$= \left|\langle S_x^1=\tfrac{\hbar}{2}\;\; S_z^2=\tfrac{\hbar}{2}|\Psi\rangle\right|^2 + \left|\langle S_x^1=\tfrac{\hbar}{2}\;\; S_z^2=-\tfrac{\hbar}{2}|\Psi\rangle\right|^2$$

Using $|S_x=\frac{\hbar}{2}\;\; S_z=\frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle+|\downarrow\rangle)\otimes|\uparrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\uparrow\rangle)$

$|S_x=\frac{\hbar}{2}\;\; S_z=-\frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle+|\downarrow\rangle)\otimes|\downarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\downarrow\rangle)$

We get $\left|\langle S_x^1=+\frac{\hbar}{2}\;\; S_z=\frac{\hbar}{2}|\Psi(T)\rangle\right|^2 = \left|\frac{1}{2}\left(e^{iT\gamma\hbar/4} + e^{-iT\gamma\hbar/4}\right)\right|^2 = \cos^2\left(\frac{T\gamma\hbar}{4}\right)$

$\left|\langle S_y^1=+\frac{\hbar}{2}\;\; S_z=-\frac{\hbar}{2}|\Psi(T)\rangle\right|^2 = 0$ ← we could have avoided some of this by noticing that the second spin is always ↑.

So $P = \cos^2\left(\frac{T\gamma\hbar}{4}\right)$

POSSIBLY USEFUL FORMULAE:

$$a = \sqrt{\frac{m\omega}{2\hbar}}\, x + ip \cdot \frac{1}{\sqrt{2m\hbar\omega}}$$

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle$$

$$a^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$a |n\rangle = \sqrt{n}\, |n-1\rangle$$

$$x = \sqrt{\frac{\hbar}{m\omega}} (a + a^\dagger)$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad S_i = \frac{\hbar}{2} \sigma_i$$

$$L_\pm = L_x \pm i L_y \qquad p = -i \sqrt{\frac{\hbar m\omega}{2}} (a - a^\dagger)$$

$$E = mc^2 \qquad \vec{F} = m\vec{a}$$

$$L_+ |\ell\ m\rangle = \hbar \sqrt{\ell(\ell+1) - m(m+1)}\, |\ell\ m+1\rangle$$

$$L_- |\ell\ m\rangle = \hbar \sqrt{\ell(\ell+1) - m(m-1)}\, |\ell\ m-1\rangle$$

$$E_0 = -\left[\frac{m}{2\hbar^2} \left(\frac{Ze^2}{4\pi\varepsilon_0} \right)^2 \right]$$

$$E = F + V - 2 \qquad E = -\vec{\mu} \cdot \vec{B}$$

$$e^{i\pi} = -1 \qquad \pi \approx 3$$

$$p = \sqrt{2m(E - V)}$$

$$c_b = -\frac{i}{\hbar} \int_0^t H'_{ba}(t') e^{i\omega_0 t'} dt'$$

$$\psi = \frac{C}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar} \int p(x) dx}$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$\vec{\mu} = \frac{gQ\vec{S}}{2m}$$

$$\sum_m \frac{|\langle \psi_m^0 | H' | \psi_n \rangle|^2}{E_n^0 - E_m^0}$$

$$\frac{m^2}{4\pi^2 \hbar^4} \left| \tilde{V}(\vec{q}) \right|^2$$

$$P_{a\to b} = \frac{|V_{ab}|^2}{\hbar^2} \frac{\sin^2[(\omega_0 - \omega) t/2]}{(\omega_0 - \omega)^2}$$

$$e^{i \frac{q}{\hbar} \oint \vec{A} \cdot d\ell}$$

$$11 + 0 + 3 = \binom{11}{3}$$

$$\psi_n = \sum_{m \neq n} \frac{\langle \psi_m^0 | H' | \psi_n^0 \rangle}{(E_n^0 - E_m^0)} \psi_m^0$$

$$H = \frac{1}{2m} (\vec{p} - \vec{A} q)^2 + q\phi$$

$$E_{nj} = mc^2 \left\{ \left[1 + \left(\frac{\alpha}{n - (j + \frac{1}{2}) + \sqrt{(j + \frac{1}{2})^2 - \alpha^2}} \right)^2 \right]^{-\frac{1}{2}} - 1 \right\}$$