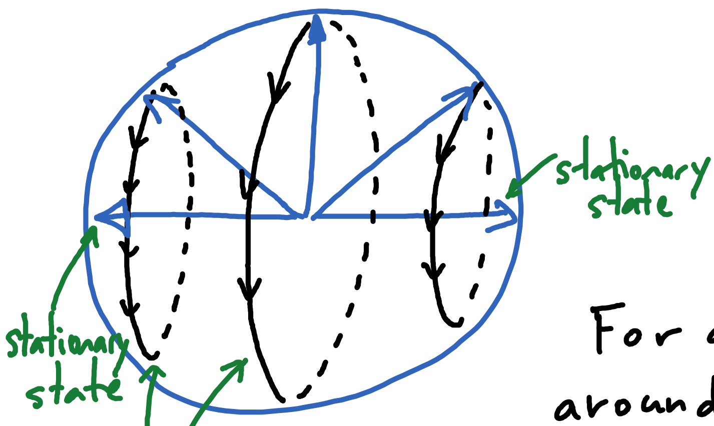
LAST TIME:

e.g. time evolution with $H = E_0 \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$



e.g. electron in magnetic field in $\hat{x}$ direction + constant potential

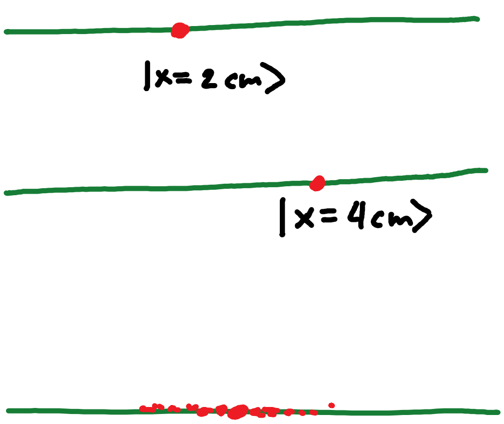
$$\hat{H} = 3E_0 \mathbb{1} + \frac{2E_0}{\hbar} \hat{S}_x$$

For any initial state, $|\Psi(t)\rangle$ rotates around x axis.

$[\hat{H}, \hat{S}_x] = 0 \Rightarrow$ rotational symmetry about x axis

$\Rightarrow \hat{S}_x$ conserved.

1D particle system: $|x\rangle$: eigenstate of position operator $\hat{X}$



General state:

$$|\Psi\rangle = \int_{-\infty}^{\infty} dx\, \psi(x) |x\rangle$$

coeff. in superposition

= WAVEFUNCTION $\psi(x) = \langle x | \psi \rangle$

Q: What is $\langle \Psi_2 | \Psi_1 \rangle$ for states w. wavefns. $\psi_1(x), \psi_2(x)$?

A: $\langle \Psi_2 | \Psi_1 \rangle = \int_{-\infty}^{\infty} dx\, \psi_1(x) \langle \Psi_2 | x \rangle = \int_{-\infty}^{\infty} dx\, \psi_2^*(x) \psi_1(x)$

$$\langle \Psi | \Psi \rangle = \int_{-\infty}^{\infty} dx\, |\psi(x)|^2$$

Hilbert space: all states where this is $< \infty$

(mathematicians call this L_2)

$\hat{\tau}(a)$: translates by a to the right

Infinitesimal: $\hat{\tau}(\varepsilon) = \mathbb{1} - i\frac{\varepsilon}{\hbar}\hat{P}$ MOMENTUM OPERATOR

Wavefunction for $\hat{\tau}(\varepsilon)|\Psi\rangle$ is $\psi(x-\varepsilon)$

$$\langle x|\left(\mathbb{1} - \frac{i\varepsilon}{\hbar}\hat{P}\right)|\Psi\rangle = \psi(x) - \varepsilon\psi'(x) + \ldots$$

$O(\varepsilon)$: $$\langle x|\hat{P}|\Psi\rangle = \frac{\hbar}{i}\frac{d}{dx}\psi'(x)$$

Acting with $\hat{P}$ on state $\longleftrightarrow$ acting with $\frac{\hbar}{i}\frac{d}{dx}$ on wave function

Q: A state has wavefunction $\psi(x)$. What is the expectation value for a measurement of momentum?

A: $\langle P\rangle = \langle\Psi|\hat{P}|\Psi\rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x)\cdot\frac{\hbar}{i}\psi(x)$

Wavefn for $\hat{X}|\Psi\rangle$ is $\langle x|\hat{X}|\Psi\rangle = x\langle x|\Psi\rangle = x\psi(x)$

Acting with $\hat{X}$ on state $\longleftrightarrow$ multiplying wavefn. by x

CONSEQUENCE: $[\hat{X},\hat{P}] = i\hbar$

By generalized uncertainty principle:

$$\Delta x\,\Delta p \geq \left|\left\langle\frac{1}{2i}[\hat{X},\hat{P}]\right\rangle\right| = \frac{\hbar}{2}$$

HEISENBERG UNCERTAINTY PRINCIPLE

Momentum operator also has eigenstates $|p\rangle$ with

$$\hat{P}|p\rangle = p|p\rangle \qquad \langle p|q\rangle = \delta(p-q)$$

Can represent any state as

$$|\Psi\rangle = \int_{-\infty}^{\infty} dp\, \psi(p)\, |p\rangle$$

where $\psi(p) = \langle p|\psi\rangle$ is the MOMENTUM WAVEFUNCTION

HOMEWORK: relation between $\psi(p)$ and $\psi(x)$,
wavefunction for $|p\rangle$

example: time evolution in 1D system.

For classical 1D system, energy is

$$E = \frac{p^2}{2m} + V(x) \qquad (*)$$

In quantum version, energy operator should be

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$$

guarantees that (*) holds for expectation values

∴ Time evolution is

$$i\hbar \frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle$$

$$\Rightarrow i\hbar \frac{d}{dt}\langle x|\psi\rangle = \langle x|\frac{\hat{p}^2}{2m} + V(\hat{x})|\psi\rangle$$

$$= \frac{1}{2m}\langle x|\hat{p}\cdot\hat{p}|\psi\rangle + \langle x|V(\hat{x})|\psi\rangle$$

$$= \frac{1}{2m}\cdot\left(\frac{\hbar}{i}\frac{d}{dx}\langle x|\hat{p}|\psi\rangle\right) + \langle x|V(x)|\psi\rangle$$

$$= \frac{1}{2m}\left(\frac{\hbar}{i}\frac{d}{dx}\right)\left(\frac{\hbar}{i}\frac{d}{dx}\right)\langle x|\psi\rangle + V(x)\langle x|\psi\rangle$$

$$\Rightarrow i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi(x)$$

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