LAST TIME: ① Symmetries ⟷ conservation laws

Given quantum system with Hamiltonian $\hat{H}$:

$$[\hat{O}, \hat{H}] = 0$$

$\swarrow$ $\hat{O}$ is conserved

$\searrow$ $\hat{T}(a) = e^{ia\hat{O}}$ is a symmetry

② Solving the Schrödinger equation

$$i\hbar \frac{d}{dt}|\Psi(t)\rangle = \hat{H}\,|\psi(t)\rangle$$

$$\Rightarrow \quad |\Psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}\,|\Psi(0)\rangle$$

If $|\Psi(0)\rangle = |E_n\rangle$:

$$|\Psi(t)\rangle = e^{-\frac{i}{\hbar}E_n t}\,|E_n\rangle$$ physically equivalent

Use linearity to solve in general:

1. Find eigenstates $|E_n\rangle$ of $\hat{H}$
2. Write $|\Psi(0)\rangle = \sum_n c_n |E_n\rangle$ $\quad c_n = \langle E_n|\Psi(0)\rangle$
3. Then: $|\Psi(t)\rangle = \sum_n c_n e^{-\frac{i}{\hbar}E_n t}|E_n\rangle$

Systems with an ∞ dimensional Hilbert space (e.g. free particle)

Here, some observables have DISCRETE SPECTRUM = set of eigenvalues

e.g. energy for particle in ∞ square well

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \quad , \quad c_n = \langle n|\psi\rangle \quad , \quad \langle n|m\rangle = \delta_{n,m}$$

just like finite dimension case

Some observables have CONTINUOUS SPECTRUM

e.g. energy for free particle, position

Dirac delta function

$$|\psi\rangle = \int dx\, \psi(x) \cdot |x\rangle \quad , \quad \psi(x) = \langle x|\psi\rangle, \quad \langle x|y\rangle = \delta(x-y)$$

wavefunction:

eigenstate of position operator: not a legitimate state, but mathematically convenient

$|\psi(x)|^2 = |\langle x|\psi\rangle|^2 =$ Probability density for measuring particle at x.

Legit states have $\int_{-\infty}^{\infty} |\psi(x)|^2 dx < \infty$

Math people call the space of such states/wavefns L^2

Third possibility: discrete + continuous spectrum.

e.g. energy for finite square well

$$|\psi\rangle = \sum c_n |E_n\rangle + \int_{E>E_1} dE\; c(E) \cdot |E\rangle$$

Important example of symmetries & conservation laws:
TRANSLATIONS & MOMENTUM.

Define: $\hat{\mathcal{T}}(a)$ translates by a to the right.

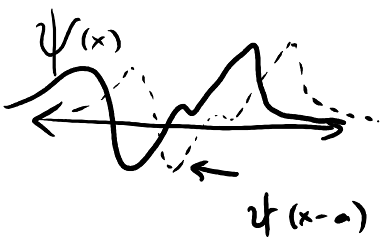
Infinitesimal: $\hat{\mathcal{T}}(\varepsilon) = \mathbb{1} - i\varepsilon \cdot \frac{1}{\hbar} \cdot \hat{P}$ chosen so P has ordinary units of momentum

$\hat{P}$ is conserved in a translation invariant system

call this MOMENTUM

e.g. 1D system:

Worksheet: if $\langle x|\psi\rangle = \psi(x)$ then



$$\langle x|\hat{\mathcal{T}}(a)|\psi\rangle = \psi(x-a)$$