LAST TIME:

Time evolution of a state is governed by the Schrödinger equation:

$$i\hbar \frac{d}{dt}|\Psi\rangle = \hat{H}|\Psi\rangle$$

$\hat{H}$ = Hamiltonian → Hermitian operator.

Conseved quantities in QM:

θ conserved $\equiv$ $\frac{d}{dt}\langle\theta\rangle = 0$ or $\frac{d}{dt}P(\lambda_n) = 0$ for all states

$\Updownarrow$

$[\hat{\theta}, \hat{H}] = 0$

example: $[\hat{H}, \hat{H}] = 0$ so H itself corresponds to a conserved quantity. This is ENERGY

TODAY: symmetries

What is a symmetry in quantum mechanics?

classical: transformation that takes solutions of equations of motion to solutions of equations of motion.

quantum: takes solutions of Schrödinger equation to solutions of Schrödinger equation.

Suppose $|\Psi(t)\rangle$ solves S.E.:

$$\frac{d}{dt}|\Psi(t)\rangle = -\frac{i}{\hbar}\hat{H}\,|\Psi(t)\rangle$$

Check if $\hat{\tau}\,|\Psi(t)\rangle$ is a solution:

$$\frac{d}{dt}\left(\hat{\tau}\,|\Psi(t)\rangle\right) = -\frac{i}{\hbar}\hat{H}\,\hat{\tau}\,|\Psi(t)\rangle$$

$$\Leftrightarrow \quad \hat{\tau}\frac{d}{dt}|\Psi(t)\rangle = -\frac{i}{\hbar}\hat{H}\hat{\tau}\,|\Psi(t)\rangle$$

$$\Leftrightarrow -\hat{\tau}\,\frac{i}{\hbar}\hat{H}\,|\Psi(t)\rangle = -\frac{i}{\hbar}\hat{H}\hat{\tau}\,|\Psi(t)\rangle$$

$$\Leftrightarrow \quad [\hat{\tau},\hat{H}]\,|\Psi(t)\rangle = 0$$

True for any solution if and only if $[\hat{\tau},\hat{H}]=0$.

For $\hat{\tau}(\varepsilon) = \mathbb{1} + i\hat{O}\varepsilon + \dots$, have also $[\hat{O},\hat{H}]=0$

Summary: $[\hat{O},\hat{H}] = 0$

$\hat{O}$ is a conserved quantity

$|\Psi\rangle \Rightarrow (\mathbb{1} + i\varepsilon\hat{O})\,|\Psi\rangle$ is infinitesimal symmetry

$|\Psi\rangle \Rightarrow e^{ia\hat{O}}\,|\Psi\rangle$ is a symmetry

SYMMETRIES $\Longleftrightarrow$ CONSERVED QUANTITIES

ASIDE: infinitesimal and finite physical transformations

$$|\Psi\rangle \to (\mathbb{1} + i\varepsilon\hat{O})|\Psi\rangle$$

repeat many times:

$$\lim_{N\to\infty}\left(\mathbb{1} + i\frac{a}{N}\hat{O}\right)^N = e^{ia\hat{O}}$$

$\hat{\tau}(a) = e^{ia\hat{O}}$ is the unitary operator corresponding to the family of physical transformations associated with $\hat{O}$

Consider Hermitian $\hat{O}$, physical transformation

$$|\Psi\rangle \rightarrow e^{ia\hat{O}}|\Psi\rangle$$

Is this a symmetry?

arbitrary initial state

solution of S.E. with this initial state

$$\begin{array}{ccc} |\Psi(0)\rangle & \longrightarrow & e^{-\frac{i}{\hbar}\hat{H}t}|\Psi(0)\rangle \\ \downarrow & & \downarrow \\ & & Ⓐ\ e^{ia\hat{O}}\left[e^{-\frac{i}{\hbar}\hat{H}t}|\Psi(0)\rangle\right] \\ e^{ia\hat{O}}|\Psi(0)\rangle & \longrightarrow & Ⓑ\ e^{-\frac{i}{\hbar}\hat{H}t}\left[e^{ia\hat{O}}|\Psi(0)\rangle\right] \end{array}$$

transformed solution

transformed initial state

actual solution of S.E. with transformed initial state

Symmetry $\Rightarrow$ transformation applied to solution of S.E. gives proper solution, i.e. Ⓐ = Ⓑ

This will be true if $[\hat{O}, \hat{H}] = 0$

same as $\hat{O}\hat{H} = \hat{H}\hat{O}$

Also: Ⓐ=Ⓑ for infinitesimal t and a gives

$$(1 + i\varepsilon\hat{O})\left(1 - i\frac{\delta t}{\hbar}\hat{H}\right)|\Psi(0)\rangle = \left(1 - i\frac{\delta t}{\hbar}\hat{H}\right)(1 + i\varepsilon\hat{O})|\Psi(0)\rangle$$

$$\Rightarrow \hat{O}\hat{H}|\Psi(0)\rangle = \hat{H}\hat{O}|\Psi(0)\rangle \rightarrow [\hat{O}, \hat{H}]|\Psi(0)\rangle = 0$$

Holds for any state only if $[\hat{O}, \hat{H}] = 0$.

★★ $|\Psi\rangle \rightarrow e^{ia\hat{O}}|\Psi\rangle$ is a symmetry if and only if $[\hat{O}, \hat{H}] = 0$ ★★

Solving the S.E.:

S.E. is equivalent to the statement that infinitesimal time evolution is $|\Psi\rangle \to (\mathbb{1} - i\frac{\delta t}{\hbar}\hat{H})|\Psi\rangle$

For $\hat{H}$ time-independent, we just keep repeating this to get $|\Psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\Psi(0)\rangle$

unitary time-evolution operator.

For eigenstate $|E_n\rangle$ of $\hat{H}$ (energy eigenstate), we get:

$$|\Psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|E_n\rangle$$

$$= e^{-\frac{i}{\hbar}E_n t}|E_n\rangle$$

state only multiplied by phase

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STATIONARY STATE

all probabilities unchanging in time