LAST TIME:

① Physical transformations act as UNITARY operations on Hilbert space:

$$|\Psi\rangle \to \mathcal{U}|\Psi\rangle \qquad \mathcal{U}^\dagger \mathcal{U} = \mathbb{1}$$

② Connection between physical transformations & observables in Q.M:

Observable $\mathcal{O}$

→ have basis $|\lambda_n\rangle$ of eigenstates, eigenvalues λ_n

$\updownarrow$

Hermitian operator $\hat{\mathcal{O}}$:

$$\hat{\mathcal{O}}|\lambda_n\rangle = \lambda_n|\lambda_n\rangle$$

$\updownarrow$

Infinitesimal transformation

$$\hat{T}(\varepsilon) = \mathbb{1} + i\varepsilon\hat{\mathcal{O}} + \mathcal{O}(\varepsilon^2)$$

$\updownarrow$

Finite transformation

repeat many times to get $\hat{T}(a)$ $\qquad a = \varepsilon N$

$$\hat{T}(a) = \lim_{N\to\infty}\left(\mathbb{1} + i\frac{a}{N}\hat{\mathcal{O}}\right)^N = e^{ia\hat{\mathcal{O}}}$$ unitary

example: Time Evolution

Have $|\Psi(t)\rangle = \hat{\tau}(t,t_0)\,|\Psi(t_0)\rangle$ initial time

for some unitary $\hat{\tau}(t,t_0)$ with $\hat{U}(t_0,t_0) = \mathbb{1}$.

Infinitesimal version:

$$|\Psi(t_0+\varepsilon)\rangle = \hat{\tau}(t_0+\varepsilon, t_0)\,|\Psi(t_0)\rangle$$

$$|\Psi(t_0+\varepsilon)\rangle = \left(\mathbb{1} + i\varepsilon \hat{B}(t_0) + \dots\right)|\Psi(t_0)\rangle$$

$$\Rightarrow \frac{|\Psi(t_0+\varepsilon)\rangle - |\Psi(t_0)\rangle}{\varepsilon} = i\,\hat{B}(t_0)\,|\Psi(t_0)\rangle$$

$$\Rightarrow \frac{1}{i}\frac{d}{dt}|\Psi(t)\rangle = \hat{B}(t)\,|\Psi(t)\rangle$$

General form of Schrödinger equation.

Call $\hat{H} = -\hbar\hat{B}$ the HAMILTONIAN: $\boxed{i\hbar\frac{d}{dt}|\Psi\rangle = \hat{H}\,|\Psi\rangle}$

time independent in simple cases.

What is a conserved quantity in QM?

Observables don't have definite values for general states, but could demand that $\langle\psi|\hat{O}|\psi\rangle$ is time independent or that $p_n = |\langle\lambda_n|\psi\rangle|^2$ is time independent.

For which $\hat{O}$ is $\langle \hat{O} \rangle$ time-independent?

Use: $\frac{d}{dt}|\psi\rangle = -\frac{i}{\hbar}\hat{H}|\psi\rangle$

$$\frac{d}{dt}\langle\psi|\hat{O}|\psi\rangle = \left(\frac{d}{dt}\langle\psi|\right)\hat{O}|\psi\rangle + \langle\psi|\hat{O}\frac{d}{dt}|\psi\rangle$$

$$= \langle\psi|\left(-\frac{i}{\hbar}\hat{H}\right)^{\dagger}\hat{O}|\psi\rangle + \langle\psi|\hat{O}\left(-\frac{i}{\hbar}\hat{H}\right)|\psi\rangle$$

$$= \frac{i}{\hbar}\langle\psi|\hat{H}\hat{O}|\psi\rangle - \frac{i}{\hbar}\langle\psi|\hat{O}\hat{H}|\psi\rangle$$

$$= \frac{i}{\hbar}\langle\psi|[\hat{H},\hat{O}]|\psi\rangle$$

zero for all states if and only if $[\hat{H},\hat{O}] = 0$

proof: use that $i[\hat{H},\hat{O}]$ is Hermitian and work in a basis where this is diagonal

example: $[\hat{H},\hat{H}] = 0$

Observable associated with time evolution is always conserved (assuming $\frac{d}{dt}\hat{H} = 0$). We call this ENERGY.