LAST TIME: Operators: linear map states $\longrightarrow$ states

Given a basis $|\lambda_n\rangle$ can represent $\hat{a}$ by matrix $a_{mn} = \langle \lambda_m | \hat{a} | \lambda_n \rangle$

Observable $\mathcal{O}$ $\longleftrightarrow$ Hermitian Operator $\hat{\mathcal{O}} |\lambda_n\rangle = \lambda_n |\lambda_n\rangle$

eigenstate of $\mathcal{O}$

For special pairs of observables $\mathcal{a}$ and $\mathcal{B}$, there is a single basis associated w. both $|a_n, b_n\rangle$

- state can have definite value for $\mathcal{a}$ & $\mathcal{B}$
- can measure $\mathcal{a}$ and $\mathcal{B}$ at same time
- $\mathcal{a}_{mn}$ and $\mathcal{B}_{mn}$ both diagonal in this basis
- $\hat{a}\hat{B}|\psi\rangle = \hat{B}\hat{a}|\psi\rangle$: the operators commute

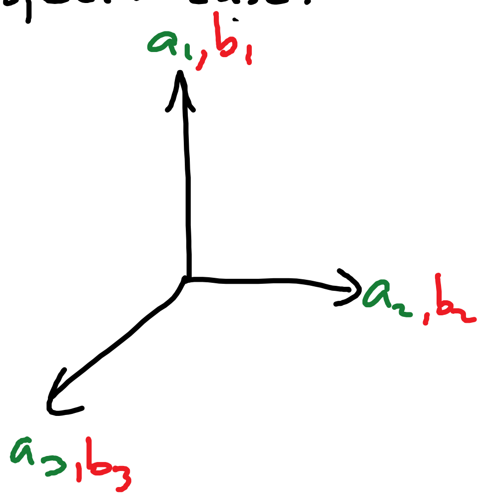
For most pairs, there is no such basis.

$$|a_n\rangle = \sum c_{nm} |b_n\rangle$$

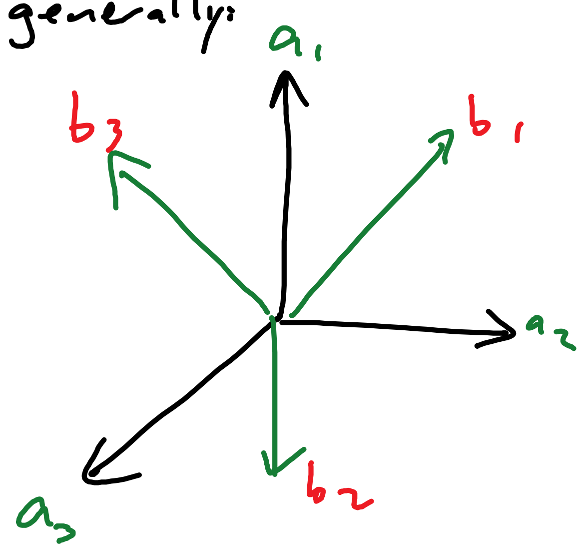
- cannot measure both at same time.

- Operators $\hat{a}$ and $\hat{B}$ don't commute $[\hat{a}, \hat{B}] \equiv \hat{a}\hat{B} - \hat{B}\hat{a} \neq 0$

special case:



generally:



Q: What is the physical meaning of $\hat{O}|\psi\rangle$ for some observable O, state $|\psi\rangle$???

Detour: How do we describe physical transformations e.g. time translation, rotation, etc... that can act as symmetries? Must obey:

① SUPERPOSITION

Should be linear maps since we want

e.g. rotated $(|\psi_1\rangle + |\psi_2\rangle)$ = rotated $|\psi_1\rangle$ + rotated $|\psi_2\rangle$

$\therefore$ should be a linear operator associated with every physical transformation $|\psi\rangle \to \hat{\tau}|\psi\rangle$

② PROBABILITY CONSERVATION

If $|\psi_1'\rangle = \hat{\tau}|\psi_1\rangle \quad |\psi_2'\rangle = \hat{\tau}|\psi_2\rangle$

want $\langle\psi_1|\psi_2\rangle = \langle\psi_1'|\psi_2'\rangle$

$= \langle\psi_1|\hat{\tau}^\dagger\hat{\tau}|\psi_2\rangle$

e.g. rotated versions of $|\psi_1\rangle, |\psi_2\rangle$

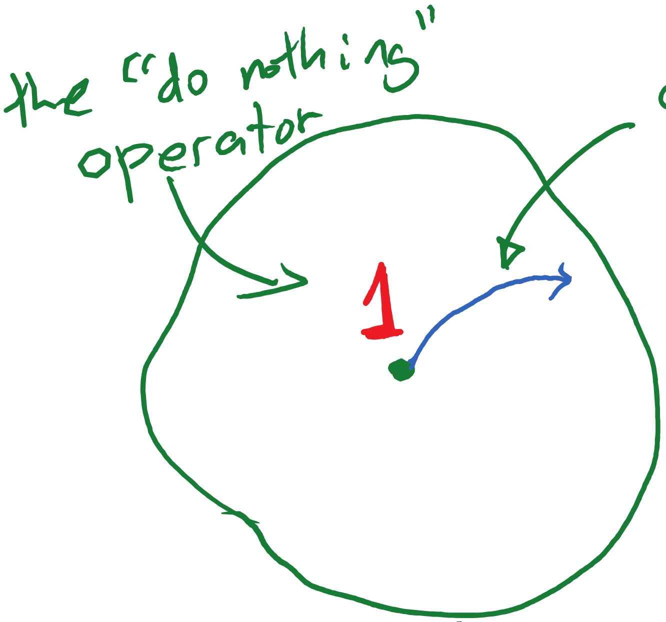
This will be true if and only if

$\hat{\tau}^\dagger\hat{\tau} = \mathbb{1} \Rightarrow \hat{\tau}$ is UNITARY

τ matrix elements are unitary matrix

⋆ Physical transformations associated with UNITARY operators ⋆

Basic property: preserves norm & inner product ∴ like <u>rotations</u> on a real vector space.



The set of all possible symmetry operators.

Some family of operations labeled by a parameter: $\hat{\tau}(\varepsilon)$
e.g. z rotations by an angle θ

$$\hat{\tau}(0) = \mathbb{1}$$

small ε:

$$\hat{\tau}(\varepsilon) = \mathbb{1} + \varepsilon \hat{B} + \ldots$$

What properties does $\hat{B}$ obey?

$$\hat{\tau}^\dagger \hat{\tau} = \mathbb{1} \Rightarrow (\mathbb{1} + \varepsilon \hat{B}^\dagger + \ldots)(\mathbb{1} + \varepsilon \hat{B} + \ldots) = \mathbb{1}$$

$$\Rightarrow \mathbb{1} + \varepsilon(\hat{B}^\dagger + \hat{B}) + \ldots = \mathbb{1}$$

$O(\varepsilon)$: $\hat{B}^\dagger = -\hat{B}$

equivalent: write $\hat{B} = i\hat{a}$. Then: $\hat{a}^\dagger = \hat{a}$

Summary: physical transformations act as unitary operators on Hilbert space.

Infinitesimal physical transformations can be represented as:

$$\hat{\tau} = 1 + i\varepsilon\hat{a} + \mathcal{O}(\varepsilon^2)$$

where $\hat{a}$ is Hermitian!

act with this many times to get a large transform.

There is a 1-1 correspondence between

infinitesimal transformations and physical observables

reminiscent of connection between symmetries & conservation laws:

translations $\longleftrightarrow$ momentum
rotations $\longleftrightarrow$ angular momentum
time translations $\longleftrightarrow$ energy