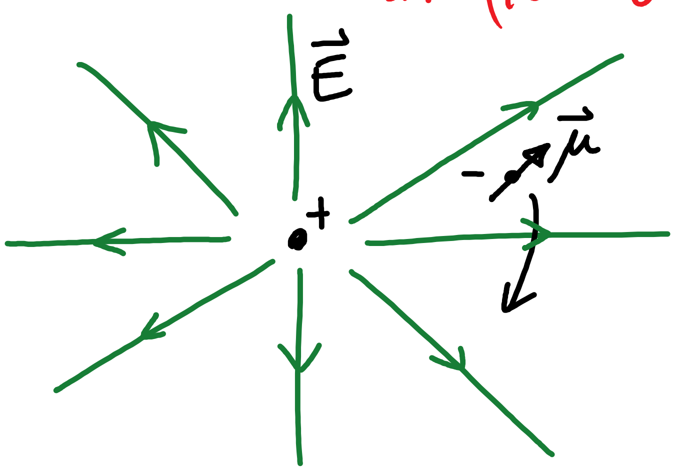
LAST TIME: H-atom including electron spin

One basis of states:

$|n\ \ell\ m\rangle \otimes |s_z\rangle$ eigenstates of H_0, L^2, L_z, S_z

Full Hamiltonian includes extra interaction

$$H_{s.o.} = \left(\frac{ge^2}{16\pi\varepsilon_0}\frac{1}{m^2c^2}\right)\cdot\frac{1}{r^3}\cdot\vec{S}\cdot\vec{L}$$



from moving magnetic dipole in electric field.

To find effects of $H_{s.o.}$, use degenerate perturbation theory: want basis where $H_{s.o.}$ is diagonal on subspace of states with same n.

Trick: define $\vec{J} = \vec{L} + \vec{S}$ total angular momentum ops.

$$\vec{S}\cdot\vec{L} = \tfrac{1}{2}\left[(\vec{L}+\vec{S})^2 - L^2 - S^2\right]$$
$$= \tfrac{1}{2}\left[J^2 - L^2 - S^2\right]$$

Eigenstates of J^2, L^2, S^2 will be "good" basis.

Example (worksheet): $n = 2, \ell = 1$

m	S_z	J_z
$\lvert 1\rangle \otimes \lvert \frac{1}{2}\rangle$		$\frac{3}{2}$
$\lvert 1\rangle \otimes \lvert -\frac{1}{2}\rangle$		$\frac{1}{2}$
$\lvert 0\rangle \otimes \lvert \frac{1}{2}\rangle$		$\frac{1}{2}$
$\lvert 0\rangle \otimes \lvert -\frac{1}{2}\rangle$		$-\frac{1}{2}$
$\lvert -1\rangle \otimes \lvert \frac{1}{2}\rangle$		$-\frac{1}{2}$
$\lvert -1\rangle \otimes \lvert -\frac{1}{2}\rangle$		$-\frac{3}{2}$

eigenstates of L^2, L_z, S^2, S_z

also: eigenstates of $J_z = L_z + S_z$

NOT necessarily eigenstates of J^2 (doesn't commute w. L_z, S_z)

We know that can split states into groups w definite J^2, i.e. there is a basis of eigenstates of J^2, J_z (generally, can't also be L_z, S_z eigenstates since these don't commute w. J^2)

Looking at the J_z values suggest states with spin $\frac{3}{2}$ $J_z = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ + remaining states w. spin $\frac{1}{2}$.

easy: $\lvert J=\frac{3}{2}, M=\frac{3}{2}\rangle = \lvert m=1\rangle \otimes \lvert S_z=\frac{1}{2}\rangle$

- act with J_- to find other $J=\frac{3}{2}$ states

e.g. $J_- \lvert J=\frac{3}{2}, M=\frac{3}{2}\rangle = (L_- + S_-)\lvert m=1\rangle \otimes \lvert S_z=\frac{1}{2}\rangle)$

$$\Rightarrow \lvert J=\tfrac{3}{2}, M=\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\lvert m=0\rangle \otimes \lvert S_z=\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\lvert m=1\rangle \otimes \lvert S_z=-\tfrac{1}{2}\rangle$$

- use orthogonality to find $\lvert J=\frac{1}{2}, M=\frac{1}{2}\rangle$:

$$\lvert J=\tfrac{1}{2}, M=\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\lvert m=0\rangle \otimes \lvert S_z=\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\lvert m=1\rangle \otimes \lvert S_z=-\tfrac{1}{2}\rangle$$

- act w. J_- to get $\lvert J=\frac{1}{2}, M=-\frac{1}{2}\rangle$

General story: for 2-part system with $\vec{J} = \vec{J}_1 + \vec{J}_2$, states $|J\,M\rangle$ with definite J^2, J_z can be written as

$$|J\,M\rangle = \sum_{m_1+m_2=M} C^{j_1 j_2 J}_{m_1 m_2 M} \, |j_1\, m_1\rangle \otimes |j_2\, m_2\rangle$$

"Clebsch-Gordon coefficients" – derive via method on worksheet or look up in table.

Allowed values of J are $j_1+j_2, j_1+j_2-1, \ldots, |j_1-j_2|$.

Same coefficients allow us to go the other way:

$$|j_1\, m_1\rangle \otimes |j_2\, m_2\rangle = \sum_{M=m_1+m_2} C^{j_1 j_2 J}_{m_1 m_2 M} \, |J\,M\rangle$$

Math language: different spins give different IRREDUCIBLE REPRESENTATIONS of rotation group.
In combined system, we have a TENSOR PRODUCT of representations, and this can be decomposed into a sum of irreducible representations

e.g. spin $1 \otimes$ spin $\frac{1}{2} \longrightarrow$ spin $\frac{3}{2}$ + spin $\frac{1}{2}$

spin $1 \otimes$ spin $1 \longrightarrow$ spin 2 + spin 1 + spin 0

Back to perturbation theory:

For $H_1 = \frac{g e^2}{16\pi^2 \varepsilon_0 m^2 c^2} \cdot \frac{1}{r^3} \vec{S} \cdot \vec{L}$

- Energy shift for state $|n\, \ell\, J\, J_z\rangle$ is:

$$\langle n\, \ell\, J\, J_z | H_1 | n\, \ell\, J\, J_z\rangle$$

$$= \frac{g e^2}{16\pi^2 \varepsilon_0 m^2 c^2} \langle n\, \ell\, J\, J_z | \frac{1}{r^3} \cdot \frac{1}{2}(J^2 - L^2 - S^2) | n\, \ell\, J\, J_z\rangle$$

$$= \frac{g e^2 \hbar^2}{32\pi^2 \varepsilon_0 m^2 c^2} \left(J(J+1) - \ell(\ell+1) - \frac{3}{4} \right) \langle n\, \ell\, J\, J_z | \frac{1}{r^3} | n\, \ell\, J\, J_z\rangle$$

$$\int_0^\infty dr\, R_{n\ell}^*(r) \cdot \frac{1}{r^3} \cdot R_{n\ell}(r) = \frac{1}{\ell\left(\ell+\frac{1}{2}\right)(\ell+1) n^3 a^3}$$

Similar size correction from relativistic correction to electron energy: