LAST WEEK:

- Have H_0 w. eigenvectors $|\psi_n^0\rangle$, eigenvalues E_n^0.
- Add λH_1; approximate energy eigenvalues as

$$E_n(\lambda) = E_n^0 + \lambda E_n + \cdots$$

TODAY

only one eigenstate of H_0 w. energy E_n^0

ordinary perturbation theory:

$$|\psi_n(\lambda)\rangle = |\psi_n^0\rangle + \lambda |\psi_n'\rangle + \cdots$$

$$E_n' = \langle \psi_n^0 | H_1 | \psi_n^0 \rangle, \text{ etc...}$$

multiple states $|A_n\rangle$ with energy E

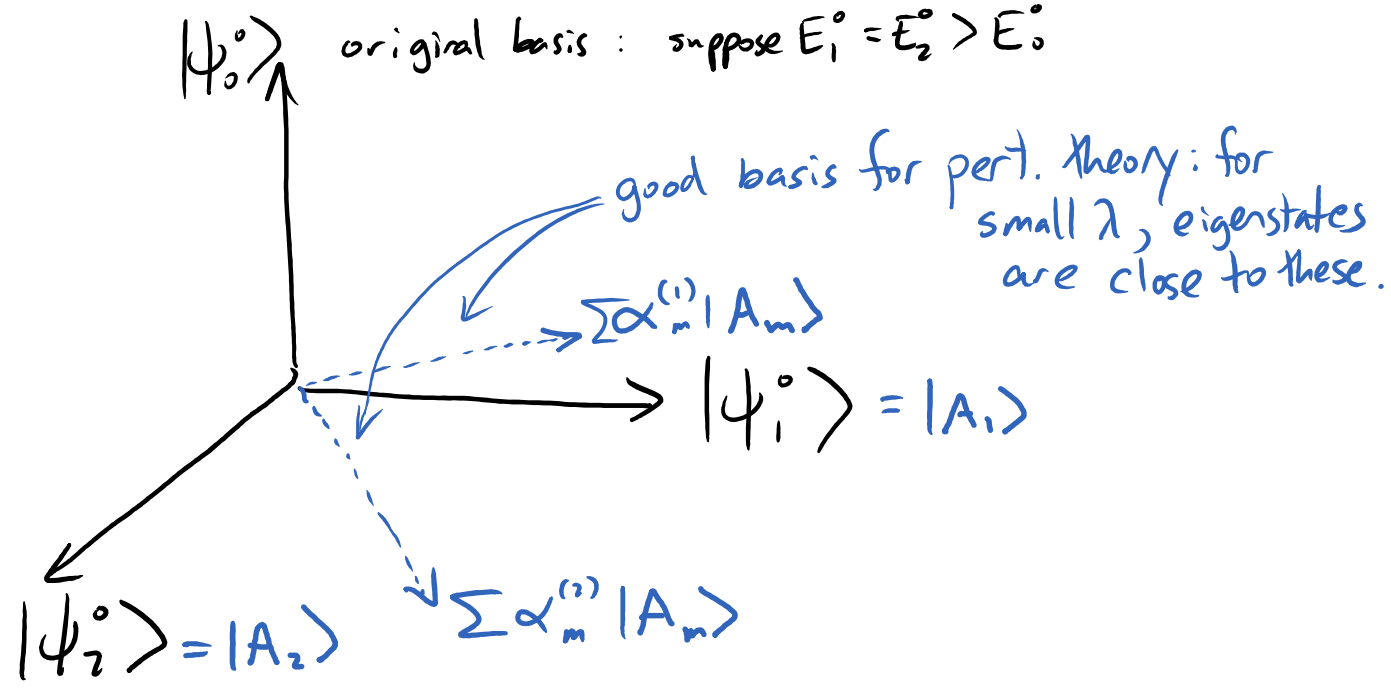
degenerate perturbation theory:

find matrix $\langle A_m | H_1 | A_n \rangle$

eigenvectors $\vec{\alpha}^{(i)}$

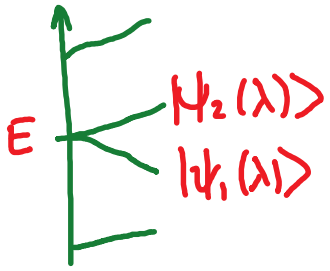
eigenvalues $\mathcal{E}^{(i)}$

Then: basis state $\sum_m \alpha_m^{(i)} |A_m\rangle$ gets shift $\lambda \mathcal{E}^{(i)}$

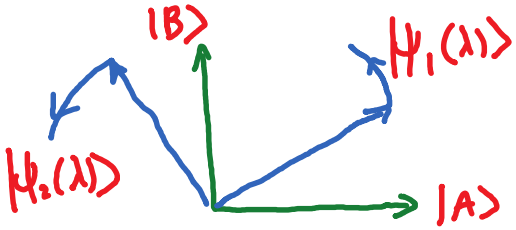


What if some of the E_n's are equal?

e.g. $|A\rangle, |B\rangle$ are both eigenstates of H_0 with same energy E. Adding perturbation λH_1 can split these:



States $|\psi_1(0)\rangle$ and $|\psi_2(0)\rangle$ are orthogonal and have energy E but are not necessarily $|A\rangle$ and $|B\rangle$: could be some linear combinations:



$\therefore$ Correct assumption:

$$|\psi_1(\lambda)\rangle = \alpha|A\rangle + \beta|B\rangle + \lambda|\delta\psi_1\rangle + \dots$$

Plug in to S.E:

$$(H_0 + \lambda H_1)(\alpha|A\rangle + \beta|B\rangle + \lambda|\delta\psi_1\rangle + \dots) = (E + \lambda\delta E_1 + \dots)(\alpha|A\rangle + \beta|B\rangle + \lambda|\delta\psi_1\rangle + \dots)$$

Order λ:

$$H_1(\alpha|A\rangle + \beta|B\rangle) = \delta E_1(\alpha|A\rangle + \beta|B\rangle) + (E - \hat{H}_0)|\delta\psi_1\rangle$$

↖ eliminates any $|A\rangle$ or $|B\rangle$ terms

Take inner product with $|A\rangle$ and $|B\rangle$

$$\alpha \langle A|H_1|A\rangle + \beta\langle A|H_1|B\rangle = \alpha\,\delta E_1$$
$$\alpha \langle B|H_1|A\rangle + \beta\langle B|H_1|B\rangle = \beta\,\delta E_1$$

$$\begin{pmatrix} \langle A|H_1|A\rangle & \langle A|H_1|B\rangle \\ \langle B|H_1|A\rangle & \langle B|H_1|B\rangle \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \delta E_1 \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

2 eigenvectors of this give correct linear combinations $|\psi_1(0)\rangle, |\psi_2(0)\rangle$

$|\psi_i(0)\rangle = \alpha|A\rangle + \beta|B\rangle$

corresponding eigenvalues give 1st order energy shift.

General method: choose a basis of eigenstates with the same energy: $|A_n\rangle$

- write the matrix elements of H_1 in this basis (only for the degenerate states)

$$\begin{pmatrix} \langle A_1|H_1|A_1\rangle & \langle A_1|H_1|A_2\rangle & \cdots \\ \langle A_2|H_1|A_1\rangle & \langle A_2|H_1|A_2\rangle & \\ \vdots & & \end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\end{pmatrix} = \delta E \begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\end{pmatrix}$$

For each eigenvector $\vec{\alpha}$, state $\alpha_1|A_1\rangle + \ldots + \alpha_n|A_n\rangle$ has energy shift δE equal to corresponding eigenvalue.

This is called DEGENERATE PERTURBATION THEORY.

WORKSHEET:

In an orthonormal basis of states $|A\rangle, |B\rangle, |C\rangle$, the Hamiltonian for a quantum system is represented as:

$$H_0 = e \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

If we add a perturbation $\lambda H_1 = \lambda \cdot \begin{pmatrix} 3 & 1 & 7 \\ 1 & 3 & 6 \\ 7 & 6 & 4 \end{pmatrix}$, write an expression for the $\lambda \to 0$ limit of the ground state (in the form $c_1|A\rangle + c_2|B\rangle + c_3|C\rangle$) and for the ground state energy to first order in λ.

Solution: the $\lambda \to 0$ eigenstates are $\alpha|A\rangle + \beta|B\rangle$ where

$$\lambda \begin{pmatrix} \langle A|H_1|A\rangle & \langle A|H_1|B\rangle \\ \langle B|H_1|A\rangle & \langle B|H_1|B\rangle \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \delta E \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\Rightarrow \quad \lambda \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \delta E \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

The eigenvectors are: $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ w. $\delta E = 4\lambda$

$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ w. $\delta E = 2\lambda$

The $\lambda \to 0$ limit of the ground state is $\frac{1}{\sqrt{2}}(|A\rangle - |B\rangle)$ with energy $e + 2\lambda$

ROTATIONS in QM

$R(\hat{n},\theta)$ = rotation by θ around $\hat{n}$

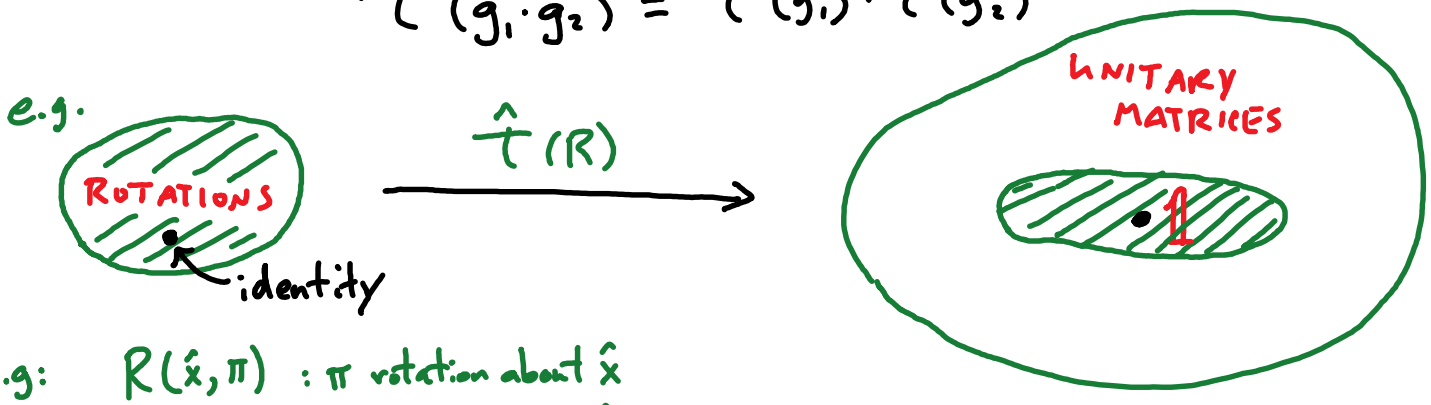
Symmetries have structure of a group:

$g_1 \cdot g_2$ = combined operation of doing g_2 then g_1

↖ associative, have identity, inverses

Action of symmetry g on a QM Hilbert space via unitary operator $\hat{T}(g)$. Must satisfy

$$\hat{T}(g_1 \cdot g_2) = \hat{T}(g_1) \cdot \hat{T}(g_2)$$



e.g: $R(\hat{x},\pi)$: π rotation about $\hat{x}$

$R(\hat{y},\pi)$: π rotation about $\hat{y}$

Have: $R(\hat{x},\pi)R(\hat{y},\pi) = R(\hat{z},\pi)$

operator for combined rotation is equal to product of operators for the individual rotations.

so: $\hat{T}(R(\hat{z},\pi)) = \hat{T}(R(\hat{x},\pi)) \cdot \hat{T}(R(\hat{y},\pi))$

any rotation: product of infinitesimal rotations

★ $\hat{T}(R)$ completely determined once we say what $\hat{T}$ is for infinitesimal rotations about x, y, z axes ★