* Perturbation theory *

General idea: if you have a problem that is "close" to one you know how to solve, often the answer is close to the answer to the solved problem.

Apply to QM. Want to find eigenstates and eigenvalues of

$$H = H_0 + \lambda H_1$$

small

Assume this has eigenstates $|\psi_n^0\rangle$ with

$$H_0 |\psi_n^0\rangle = E_n^0 |\psi_n^0\rangle$$

For small λ we expect that:

$$|\psi_n\rangle = |\psi_n^0\rangle + \lambda |\psi_n^1\rangle + \lambda^2 |\psi_n^2\rangle + \ldots$$

$$E_n = E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + \ldots$$

Plug into time indep. S.E.

$$H |\psi_n\rangle = E_n |\psi_n\rangle$$

Get:

$$(H_0 + \lambda H_1)(|\psi_n^0\rangle + \lambda |\psi_n^1\rangle + \ldots)$$

$$= (E_n^0 + \lambda E_n^1 + \ldots)(|\psi_n^0\rangle + \lambda |\psi_n^1\rangle + \ldots)$$

Write separate eqns for $\mathcal{O}(\lambda^0), \mathcal{O}(\lambda^1), \mathcal{O}(\lambda^2) \ldots$ to determine corrections.

Order λ^0 terms:

$$H_0|\psi_n^0\rangle = E_n^0|\psi_n^0\rangle$$

already assumed this

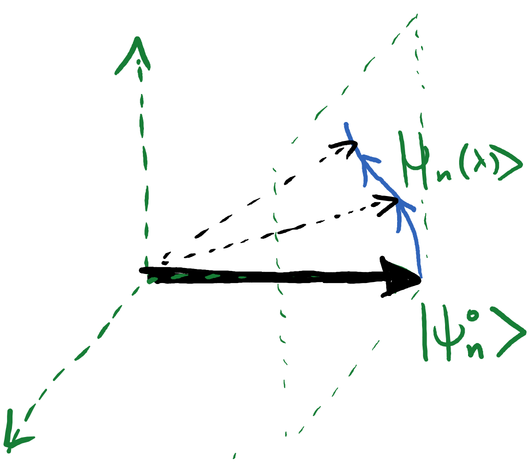
Order λ terms:

$$H_0|\psi_n^1\rangle + H_1|\psi_n^0\rangle = E_n^0|\psi_n^1\rangle + E_n^1|\psi_n^0\rangle$$

Can write $|\psi_n^1\rangle$ in terms of complete basis $|\psi_n^0\rangle$:

$$|\psi_n^1\rangle = \sum_{m\neq n} c_n^m |\psi_m^0\rangle$$

we will normalize $|\psi_n\rangle$ so that coefficient of $|\psi_n^0\rangle$ is always 1.



Get:

$$\sum_{m\neq n} c_n^m E_m^0|\psi_m^0\rangle + H_1|\psi_n^0\rangle = \sum_{m\neq n} E_n^0 c_n^m|\psi_m^0\rangle + E_n^1|\psi_n^0\rangle$$

$$\Rightarrow \quad E_n^1|\psi_n^0\rangle + \sum_{m\neq n}(E_n^0 - E_m^0)c_n^m|\psi_m^0\rangle = H_1|\psi_n^0\rangle$$

component along $|\psi_n^0\rangle$ — other components

This is a vector equation: get one equation for each component. Take inner product w. $|\psi_n^0\rangle$ to get component along original direction:

$$\boxed{E_n^1 = \langle \psi_n^0 | \hat{H}_1 | \psi_n^0 \rangle}$$

1st order correction to energy

Take inner product w. $|\psi_\ell^0\rangle$ $(\ell \neq n)$ to get other components:

$$(E_n^0 - E_\ell^0)\, c_n^\ell = \langle \psi_\ell^0 | H_1 | \psi_n^0 \rangle$$

$$\therefore \boxed{|\psi_n^1\rangle = \sum_{m \neq n} |\psi_m^0\rangle \frac{\langle \psi_m^0 | H_1 | \psi_n^0 \rangle}{E_n^0 - E_m^0}}$$

1st order correction to wavefunction

we need to be more careful if some energies are the same!

Often find $E_n^1 = 0$ so first correction to energy is at next order.

$\Theta(\lambda^2)$:

$$H_0 |\psi_n^2\rangle + H_1 |\psi_n^1\rangle = E_n^0 |\psi_n^2\rangle + E_n^1 |\psi_n^1\rangle + E_n^2 |\psi_n^0\rangle$$

Take inner product w. $|\psi_n^0\rangle$:

$$\langle \psi_n^0 | H_1 | \psi_n^1 \rangle = E_n^2$$

Use our result for $|\psi_n^1\rangle$:

$$E_n^2 = \sum_{m \neq n} \frac{\langle \psi_n^0 | H_1 | \psi_m^0 \rangle \langle \psi_m^0 | H_1 | \psi_n^0 \rangle}{E_n^0 - E_m^0}$$

$$\boxed{E_n^2 = \sum_{m \neq n} \frac{|\langle \psi_n^0 | H_1 | \psi_m^0 \rangle|^2}{E_n^0 - E_m^0}}$$

2nd order correction to energy: always -ve for ground state.