Name & Student number:

Physics 402 Exam, April 10, 2019

Multiple Choice Questions: Please write your answers in the spaces on page 4

1) For a state $|\psi\rangle$ and a Hermitian operator $\hat{B}$ associated with an observable B, suppose that $\langle\psi|\hat{B}|\psi\rangle = 3$. Then we can say that

a) The observable B has a definite value of 3 in the state $|\psi\rangle$.

b) The observable B does not necessarily have a definite value of B before measuring, but we will find the value 3 if we measure B.

c) The observable B does not necessarily have a definite value of B before measuring, but the average result for measurements of B on states identical to $|\psi\rangle$ will approach 3 in the limit where the number of measurements becomes large.

d) The observable B does not necessarily have a definite value of B; the quantity $\langle\psi|\hat{B}|\psi\rangle$ does not have any direct connection to the results of measurements of B.

2) If H is the Hermitian operator associated with the energy of a quantum system and $|\psi\rangle$ is some state of the system, then $H|\psi\rangle$ is always equal to

a) the expectation value of the energy in the state $|\psi\rangle$.

b) the energy of the state $|\psi\rangle$ if $|\psi\rangle$ is an energy eigenstate, or zero otherwise.

c) some constant times the amount by which the state changes during an infinitesimal time evolution.

d) the state $|\psi\rangle$, times the energy expectation value for this state.

3) If Hermitian operators $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ commute with each other, one consequence is that

a) The observables $\mathcal{A}$ and $\mathcal{B}$ are conserved.

b) The observables $\mathcal{A}$ and $\mathcal{B}$ are equal to each other for all states.

c) There is a basis of states with definite values of $\mathcal{A}$ that also have definite values for $\mathcal{B}$.

d) All states have definite values for $\mathcal{A}$ and $\mathcal{B}$.

4) If the state of a quantum system with a time-independent Hamiltonian is some energy eigenstate $|E\rangle$ at time $t = 0$, which of the following is necessarily true?

a) Physical observables for this system will oscillate periodically with a frequency proportional to the energy E.

b) All physical observables will be independent of time for this state.

stationary state

$|\psi(t)\rangle = e^{-iEt/\hbar}|\psi(0)\rangle$

c) The state has a definite value for all physical observables.

d) The state has a definite value for any physical observable associated with a time-independent operator.

5) Physical transformations such as rotations are represented by unitary operators acting on the Hilbert space of a quantum system. Which of the following is **not** a property of unitary operators?

a) They are linear maps from the Hilbert space to itself.

b) They preserve the inner product between states: if $|A'\rangle = U|A\rangle$ and $|B'\rangle = U|B\rangle$, then $\langle A'|B'\rangle = \langle A|B\rangle$.

c) They have an orthogonal basis of eigenvectors with real eigenvalues. this is for Hermitian ops

d) They map normalized states to normalized states.

6) Suppose we add a perturbation $H_1 = \lambda x^3$ to a harmonic oscillator. The first non-zero correction to the energy of the ground state comes at second order in perturbation theory. In the sum over states appearing in the formula for the second order energy shift, how many terms are non-zero in this case?

a) 1

b) 2

c) 3

d) 4

e) an infinite number

$x^3 = c \cdot (a + a^\dagger)^3$

$\langle n | (a + a^\dagger)^3 | 0 \rangle \neq 0$ for $n = 1, 3$

7) A spinless particle in a three-dimensional harmonic oscillator potential is in an eigenstate of the total angular momentum squared operator with eigenvalue zero. Which of the following is **not** necessarily true

a) The wavefunction for this particle will be spherically symmetric.

b) The particle is in the ground state. e.g. 2s, 3s... for H atom

c) A measurement of the z component of angular momentum will necessarily give zero.

d) Performing a rotation on this state will give back the same state.

8) A class of students participates in a contest to find the best bound for the ground state energy of a quantum system using the variational method. Jake uses an unnormalized state $|\chi\rangle$ as his ansatz for the ground state. If E_0 is the true ground state energy, which of the following is the bound that Jake obtains?

a) $\langle\chi|H|\chi\rangle \leq E_0$

b) $\langle\chi|H|\chi\rangle \geq E_0$

c) $\langle\chi|H|\chi\rangle/\langle\chi|\chi\rangle \geq E_0$

d) $\langle\chi|H|\chi\rangle/\langle\chi|\chi\rangle \leq E_0$

Normalized state

$|\chi'\rangle = \dfrac{|\chi\rangle}{\langle\chi|\chi\rangle^{1/2}}$ $\quad E_0 \leq \langle\chi'|H|\chi'\rangle$

9) For an electron in a hydrogen atom in the state $|nlJM\rangle = |3, 1, 3/2, 1/2\rangle$, which of the following is not a possible outcome for a measurement of the z component of orbital angular momentum?

$J = \frac{3}{2}$ $M = \frac{1}{2}$ is a linear combination of $m = 1$ $s_z = -\frac{1}{2}$ and $m = 0$ $s_z = \frac{1}{2}$

a) $\hbar$
b) 0
c) $-\hbar$
d) None of the above: all of these are possible outcomes.

10) In the electric dipole approximation for atomic transitions, an atom in state $|b\rangle$ can make a transition to a lower-energy state $|a\rangle$ if and only if
a) The matrix element of some component of the electric dipole operator is nonvanishing between $|b\rangle$ and $|a\rangle$.
b) There is some background radiation with frequency $\omega = |E_a - E_b|/\hbar$. (could have spontaneous transition)
c) The matrix element of some component of the electric dipole operator is nonvanishing between $|b\rangle$ and $|a\rangle$ AND there is some background radiation with frequency $\omega = |E_a - E_b|/\hbar$.
d) None of the above: dipole transitions are always possible whether or not a) or b) are true; those conditions just tell us when these transitions are most likely.

11) A quantum system consists of a spin 1 (dim 3) particle and a spin 3/2 (dim 4) particle, each at fixed location. The dimension of the total Hilbert space representing the spin states of the two particles is

dim = 3×4

a) 2 b) 3 c) 4 d) 7 e) 12

12) In the presence of a uniform magnetic field, the Hamiltonian describing the interaction between the electron's spin and the magnetic field is
a) $H_B = \frac{ge}{2m}\vec{S} \cdot \vec{B}$
b) $H_B = -\frac{e}{m}\vec{p} \times \vec{B}$
c) $H_B = e\vec{x} \cdot \vec{B}$
d) $H_B = -e^2 S^2 B^2$

$$H = -\vec{\mu} \cdot \vec{B}$$
$$\vec{\mu} = -\frac{ge}{2m}\vec{S}$$

13) Consider two non-interacting *identical* particles of spin 1/2 in the same three-dimensional harmonic oscillator potential. The ground state of this system

fermion antisymmetric under interchange (won't be covered for 2020 exam)

a) $\frac{1}{\sqrt{2}}(|000\uparrow\rangle \otimes |000\downarrow\rangle - |000\downarrow\rangle \otimes |000\uparrow\rangle)$
b) $\frac{1}{\sqrt{2}}(|000\uparrow\rangle \otimes |000\downarrow\rangle + |000\downarrow\rangle \otimes |000\uparrow\rangle)$
c) $|000\downarrow\rangle \otimes |000\downarrow\rangle$
d) $|000\uparrow\rangle \otimes |000\uparrow\rangle$
e) All of the above are valid states with the same energy.

14) For the singlet state $\frac{1}{\sqrt{2}}(|\uparrow_e\rangle \otimes |\downarrow_p\rangle - |\downarrow_e\rangle \otimes |\uparrow_p\rangle)$ of electron and proton spins in the hydrogen atom ground state, we perform a measurement of some physical quantity on the electron spin. We can say that the results of this measurement will be the same as for

a) an electron in the state $\frac{1}{\sqrt{2}}(|\uparrow_e\rangle + |\downarrow_e\rangle)$.
b) an electron in the state $\frac{1}{\sqrt{2}}(|\uparrow_e\rangle - |\downarrow_e\rangle)$.
c) an electron in the state $\frac{1}{\sqrt{2}}(|\uparrow_e\rangle + i|\downarrow_e\rangle)$.
d) an electron in the state $\frac{1}{\sqrt{2}}(|\uparrow_e\rangle - i|\downarrow_e\rangle)$.
e) an electron that is in the state $|\uparrow_e\rangle$ with probability 1/2 and in the state $|\downarrow_e\rangle$ with probability 1/2. (not covered 2020)

15) In quantum field theory, a field $\phi(x)$ defined on a spatial interval $[0, L]$ can be represented as a Fourier series $\phi(x) = \sum_n \phi_n \sin(n\pi x/L)$, where the quantities ϕ_n each behave like a harmonic oscillator. If $|0\rangle$ is the ground state of the system and $a_n^\dagger$ represents the creation operator associated with the oscillator ϕ_n, we can say that the state $(a_2^\dagger)(a_1^\dagger)^3|0\rangle$ represents

a) A state of three particles with a certain momentum and one particle with a smaller momentum.
$a_1^\dagger$: creates particle $\lambda=$
b) A state of three particles with a certain momentum and one particle with a larger momentum.
$a_2^\dagger$ creates particle $\lambda=$
c) A state of one particle with three units of energy and one particle with a single unit of energy (but a different wavelength).
d) A state of two particles with a certain energy and one particle with a larger energy.
e) A state of two particles with a certain energy and one particle with a smaller energy.

Multiple Choice Answers (1 point each):

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15

② For $\delta = 0$, we have two uncoupled harmonic oscillators, so the states are $|n_1\ n_2\rangle = |n_1\rangle \otimes |n_2\rangle$, with energy

$$E = \hbar\omega(n_1 + \tfrac{1}{2}) + \hbar\omega(n_2 + \tfrac{1}{2})$$
$$= \hbar\omega(n_1 + n_2 + 1)$$

The lowest energy state is $|0\ 0\rangle$ with energy $\hbar\omega$. We have two states at energy $2\hbar\omega$, $|0\ 1\rangle$ and $|1\ 0\rangle$.

We now add the perturbation:

$$H_1 = \delta(x_1 - x_2)^2$$
$$= \delta(x_1^2 - 2x_1x_2 + x_2^2)$$
$$= \delta \cdot \frac{\hbar}{2m\omega}\left((a_1 + a_1^\dagger)^2 - 2(a_1 + a_1^\dagger)(a_2 + a_2^\dagger) + (a_2 + a_2^\dagger)^2\right)$$

The first order shift in the ground state energy is:

$$\langle 0\ 0 | H_1 | 0\ 0\rangle = \delta \frac{\hbar}{2m\omega} \langle 0 0 | a_1 a_1^\dagger + a_2 a_2^\dagger | 0 0\rangle$$
$$= \delta \cdot \frac{\hbar}{m\omega}$$

↖ we only wrote the terms that give a non-zero result.

So the lowest energy is $\hbar\omega + \delta \cdot \frac{\hbar}{m\omega}$.

We need degenerate perturbation theory to find the energy shifts for the $E = 2\hbar\omega$ states. We have:

$$\langle 1\ 0 | H_1 | 1\ 0\rangle = \delta \frac{\hbar}{2m\omega} \langle 1 0 | a_1 a_1^\dagger + a_1^\dagger a_1 + a_2^\dagger a_2 | 1\ 0\rangle$$
$$= \delta \frac{\hbar}{2m\omega}(2 + 1 + 1) = 2\frac{\delta\hbar}{m\omega}$$

$$\langle 0\ 1 | H_1 | 0\ 1\rangle = 2\frac{\delta\hbar}{m\omega} \quad \text{(same calculation with } 1 \leftrightarrow 2\text{)}$$

$$\langle 0\ 1 | H_1 | 1\ 0\rangle = \delta \frac{\hbar}{2m\omega} \langle 1\ 0 | -2 a_1 a_2^\dagger | 1\ 0\rangle$$
$$= -\delta \frac{\hbar}{m\omega}$$

$$\langle 1\ 0 | H_1 | 0\ 1\rangle = \langle 0\ 1 | H_1 | 1\ 0\rangle^* = -\delta \frac{\hbar}{m\omega}$$

So the matrix elements of H_1 on our degenerate subspace are:

$$\frac{\delta\hbar}{m\omega}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

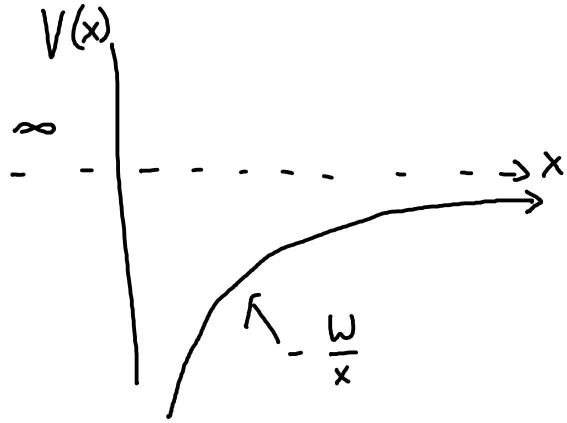
The matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ has characteristic polynomial

$$\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1).$$

So the energy shifts are $\frac{3\delta\hbar}{m\omega}$ and $\frac{\delta\hbar}{m\omega}$. The state with the second lowest energy overall therefore has energy

$$E_2 = 2\hbar\omega + \frac{\delta\hbar}{m\omega}.$$

(3)



We would like to place an upper bound on the ground state energy in the potential shown, using the variational method with trial wavefunction

$$\psi(x) = \begin{cases} 0 & x<0 \\ A\cdot x e^{-bx} & x>0 \end{cases}$$

First, we choose A to normalize $\psi(x)$. We have:

$$\int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = A^2 \int_0^{\infty} dx \cdot x^2 e^{-2bx}$$

$$= A^2 \cdot \frac{2}{(2b)^3}$$

Setting this to 1, we get that $A = 2b^{3/2}$.

Now, we have that:

$$E_0 \leq \langle\psi|H|\psi\rangle = \langle\psi|\frac{p^2}{2m} + V(x)|\psi\rangle$$

$$= \int_0^{\infty} dx\left(-\frac{\hbar^2}{2m}\psi(x)\psi''(x) + V(x)\psi^2(x)\right)$$

$$= \int_0^{\infty} dx\left(\frac{\hbar^2}{2m}(\psi'(x))^2 + V(x)\psi^2(x)\right)$$

$$= \frac{\hbar^2}{2m}A^2\int_0^{\infty} dx\left(e^{-2bx} - 2bxe^{-2bx} + b^2x^2e^{-2bx}\right)$$

$$-W\cdot A^2\int_0^{\infty} dx \cdot x \cdot e^{-2bx}$$

$$= \frac{\hbar^2}{2m}\cdot(4b^3)\left[\frac{1}{2b} - \frac{2b}{(2b)^2} + \frac{2b^2}{(2b)^3}\right] - W\cdot(2b^3)\cdot\frac{1}{(2b)^2}$$

Simplifying, we get:

$$E_0 \leqslant \frac{\hbar^2}{2m} \cdot b^2 - W \cdot b$$

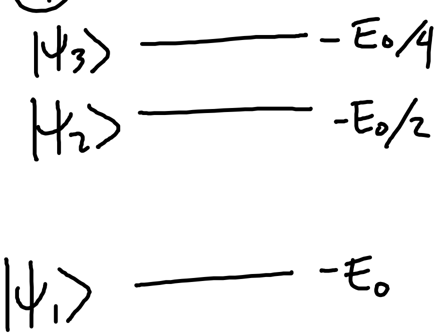
To get the best bound, we minimize over b. At the minimum, we have:

$$0 = \frac{d}{db}\left(\frac{\hbar^2}{2m} b^2 - Wb\right) = \frac{\hbar^2}{m} \cdot b - W$$

$$\Rightarrow b = \frac{mW}{\hbar^2}$$

Thus, we have: $\boxed{E_0 \leqslant -\frac{1}{2} \frac{mW^2}{\hbar^2}}$

④



Molecules in state $|\psi_2\rangle$ can make a transition to $|\psi_3\rangle$ via absorption and to $|\psi_1\rangle$ via stimulated emission or spontaneous emission. The rate for absorption $2\to3$ is:

$$R_{2\to3} = \frac{\pi}{3\varepsilon_0\hbar^2}\left(|\wp_{23}^x|^2 + |\wp_{23}^y|^2 + |\wp_{23}^z|^2\right)\cdot\rho\left(\frac{|E_3-E_2|}{\hbar}\right)$$

$$= \frac{\pi}{3\varepsilon_0\hbar^2}\cdot\left(P_0^2 + 4P_0^2 + 0\right)\cdot\rho\left(\frac{E_0}{4\hbar}\right)$$

$$= \frac{5\pi P_0^2}{3\varepsilon_0\hbar^2}\cdot\frac{E_0^3}{4\pi^2\hbar^2c^3} = \frac{5}{12}\cdot\frac{P_0^2E^3}{\pi\hbar^4c^3\varepsilon_0}$$

The rate for stimulated emission $2\to1$ is:

$$R_{2\to1} = \frac{\pi}{3\varepsilon_0\hbar^2}\left(|\wp_{21}^x|^2 + |\wp_{21}^y|^2 + |\wp_{21}^z|^2\right)\cdot\rho\left(\frac{|E_2-E_1|}{\hbar}\right)$$

$$= \frac{\pi}{3\varepsilon_0\hbar^2}\left(P_0^2\cdot(2+i)(2-i) + P_0^2\cdot i(-i) + P_0^2\cdot(-1)^2\right)\cdot\rho\left(\frac{E_0}{2\hbar}\right)$$

$$= \frac{7\pi P_0^2}{3\varepsilon_0\hbar^2}\cdot\frac{E_0^3}{8\pi^2\hbar^2c^3} = \frac{7}{24}\,\frac{P_0^2E_0^3}{\pi\hbar^4c^3\varepsilon_0}$$

The rate for spontaneous emission $2\to1$ is:

$$A_{2\to1} = \left(\frac{|E_2-E_1|}{\hbar}\right)^3\cdot\frac{1}{3\pi\varepsilon_0\hbar c^3}\cdot\left(|\wp_{21}^x|^2 + |\wp_{21}^y|^2 + |\wp_{21}^z|^2\right)$$

$$= \left(\frac{E_0}{2\hbar}\right)^3\cdot\frac{1}{3\pi\varepsilon_0\hbar c^3}\cdot\left(7P_0^2\right)$$

$$= \frac{7}{24}\,\frac{E_0^3\cdot P_0^2}{\varepsilon_0\hbar^4c^3\pi}$$

The net transition rate is then:

$$A = R_{2\to3} + R_{2\to1} + A_{2\to1}$$
$$= \left(\frac{5}{12} + \frac{7}{24} + \frac{7}{24}\right)\cdot \frac{P_0^2 E_0^3}{\pi \varepsilon_0 \hbar^4 c^3}$$
$$= \frac{P_0^2 E_0^3}{\pi \varepsilon_0 \hbar^4 c^3}$$

The number of molecules per unit time making a transition is: $N_0 \cdot A = 100{,}000 \cdot \frac{P_0^2 E_0^3}{\pi \varepsilon_0 \hbar^4 c^3}$. So the time expected for 100 atoms to make a transition is:

$$T = \frac{100}{100{,}000 \frac{P_0^2 E_0^3}{\pi \varepsilon_0 \hbar^4 c^3}} = \frac{\pi \varepsilon_0 \hbar^4 c^3}{1000 P_0^2 E_0^3}$$

⑤ The transition probability is

$$P(T) = \frac{1}{\hbar^2} \left| \int_0^T dt \, (H_1)_{ba}(t) \cdot e^{i\omega_{ba} t} \right|^2$$

where
$$\omega_{ba} = \frac{E_{J=1,M=0} - E_{J=0,M=0}}{\hbar} = \frac{1}{\hbar}\left(\frac{\Omega}{\hbar} \cdot \hbar^2 \cdot 2 \cdot 1 - 0\right) = 2\Omega$$

and
$$(H_1)_{ba}(t) = \langle J=0, M=0 | H_1(t) | J=1, M=0 \rangle$$
$$= A e^{-bt} \cdot \frac{1}{2}\left(\langle \uparrow\downarrow| - \langle \downarrow\uparrow|\right) S_2^z \left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$
$$= A e^{-bt} \cdot \frac{1}{2} \cdot \left(\langle \uparrow\downarrow| - \langle \downarrow\uparrow|\right)\left(-\frac{\hbar}{2} |\uparrow\downarrow\rangle + \frac{\hbar}{2} |\downarrow\uparrow\rangle\right)$$
$$= -\frac{A\hbar}{2} \cdot e^{-bt}$$

So:
$$P(T=\infty) = \frac{A^2}{4} \left| \int_0^\infty dt \, e^{-bt + 2i\Omega t} \right|^2$$
$$= \frac{A^2}{4} \left| \frac{1}{b - 2i\Omega} \right|^2$$
$$= \frac{A^2}{4} \frac{1}{b^2 + 4\Omega^2}$$

b) For $b = 0$, our Hamiltonian is:
$$H = \frac{\Omega}{\hbar} J^2 + 2\sqrt{3}\,\Omega \cdot S_2^z$$

Let's write the matrix elements of this for our states in the $|J \; M\rangle$ basis. We have that

$|1\ 1\rangle = |\uparrow\uparrow\rangle$

$|1\ {-1}\rangle = |\downarrow\downarrow\rangle$

$|1\ 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$

$|0\ 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

Then, using the order above

$$H = \hbar\Omega \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \sqrt{3}\hbar\Omega \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

So $|1\ 1\rangle$ and $|1\ {-1}\rangle$ are eigenstates, and the other eigenstates are

$$\alpha|1\ 0\rangle + \beta|0\ 0\rangle$$

where $\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 2 & -\sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix} \cdot \hbar\Omega$

The eigenvalues of this are

$$\lambda = 3\hbar\Omega \text{ and } \lambda = -\hbar\Omega$$

with eigenvectors $\frac{1}{2}\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}$ and $\frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$. So

$|\psi_1\rangle = -\frac{\sqrt{3}}{2}|1\ 0\rangle + \frac{1}{2}|0\ 0\rangle$ has energy $3\hbar\Omega$

$|\psi_2\rangle = \frac{1}{2}|1\ 0\rangle + \frac{\sqrt{3}}{2}|0\ 0\rangle$ has energy $-\hbar\Omega$

Finally, we have $|\psi(0)\rangle = |1\ 0\rangle = -\frac{\sqrt{3}}{2}|\psi_1\rangle + \frac{1}{2}|\psi_2\rangle$

$$\Rightarrow |\psi(T)\rangle = -\frac{\sqrt{3}}{2}e^{-3i\Omega T}|\psi_1\rangle + \frac{1}{2}e^{i\Omega T}|\psi_2\rangle$$

$$\Rightarrow P_{J=0}(T) = |\langle 0\ 0|\psi(T)\rangle|^2 = \left|-\frac{\sqrt{3}}{4}e^{-3i\Omega T} + \frac{\sqrt{3}}{4}e^{i\Omega T}\right|^2$$

$$= \frac{3}{4}\cdot\left|\frac{e^{2i\Omega T} - e^{-2i\Omega T}}{2i}\right|^2$$

$$= \frac{3}{4}\sin^2(2\Omega T)$$