15) In a simple quantum field theory associated with a classical system that obeys the wave equation, a particle can be understood as

a) one of the elementary constituents that make up the field through their collective motion

b) a pointlike object whose motion is what gives rise to oscillations in the field

c) a quantum of energy in a harmonic oscillator that describes a particular Fourier mode of the field

d) a standing wave in the field whose position is localized to some small region

16) The Bell inequalities show that

a) statistical results of experiments measuring the spins of entangled particles cannot be explained by a model in which measurement outcomes are predetermined and the measurements do not affect each other.

b) the non-deterministic behaviour of quantum mechanics cannot really be true: there must be some underlying description in which all states have definite values for physical observables.

c) quantum entanglement can be used to transmit information over large distances faster than is possible according to the rules of special relativity.

d) the bells of St. Clement's are always at least as loud as, but may be louder than the bells of St. Martin's.

Multiple Choice Answers:

1 C	2 B	3 C	4 B	5 A	
6 A	7 B	8 E	9 D	10 D	
11 B	12 B	13 D	14 E	15 C	16 A

PROBLEM 1 SOLUTION

To find the time evolution of $|\psi\rangle$, we decompose $|\psi(0)\rangle$ into a linear combination of energy eigenstates, which are each multiplied by $e^{-iEt/\hbar}$ under time evolution.

The Hamiltonian is proportional to J^2, so the energy eigenstates ~~can be~~ are eigenstates of J^2. A basis of these are the J^2, J_z eigenstates $|J\ M\rangle$. In our case, we have two spin $\frac{1}{2}$ particles so we can have $J=0$ with $M=0$ or $J=1$ with $M=\pm 1, 0$.

The initial state is an eigenstate of J_z^1 and J_z^2 with $J_z^{tot}=0$, so will be a linear combination of $|J=0\ M=0\rangle$ and $|J=1\ M=0\rangle$. From the Clebsch Gordon table, or from knowledge of the two-spin system, we have that

$$|J=0, M=0\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle - |{-\tfrac{1}{2}}\ \tfrac{1}{2}\rangle\right)$$

$$|J=1, M=0\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle + |{-\tfrac{1}{2}}\ \tfrac{1}{2}\rangle\right)$$

So our initial state is

$$|\psi(0)\rangle = |\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle = \frac{1}{\sqrt{2}}\left(|J=0, M=0\rangle + |J=1, M=0\rangle\right)$$

At time T, we have then:

$$|\psi(T)\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{iE_0 t}{\hbar}}|J=0\ M=0\rangle + e^{-\frac{iE_1 t}{\hbar}}|J=1, M=0\rangle\right)$$

where $E_0 = 0$ and $E_1 = \frac{E}{\hbar^2}\cdot\hbar^2\cdot 1\cdot 2 = 2E$. In the J_z^e, J_z^p basis, this is:

$$\frac{1}{2}\left(|\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle - |{-\tfrac{1}{2}}\ \tfrac{1}{2}\rangle\right) + \frac{1}{2}e^{-\frac{2Eti}{\hbar}}\left(|\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle + |\tfrac{1}{2}\ \tfrac{1}{2}\rangle\right)$$

So: $$P_{S_z^e=\frac{\hbar}{2}}(T) = \left|\frac{1}{2} + \frac{1}{2}e^{-\frac{2Eti}{\hbar}}\right|^2 = \left|e^{-Eti/\hbar}\cdot\left(\frac{e^{-iEt/\hbar} - e^{iEt/\hbar}}{2}\right)\right|^2 = \sin^2\left(\frac{ET}{\hbar}\right)$$

PROBLEM 2 SOLUTIONS

We have that the true ground state energy must be less than or equal to $\langle \psi(\theta) | H | \phi(\theta) \rangle$, where:

$$|\psi(\theta)\rangle = \cos\theta \, |m = \tfrac{3}{2}\rangle + \sin\theta \, |m = -\tfrac{1}{2}\rangle$$

This state is already normalized. To calculate the required matrix elements, we note that:

$$J_x = \frac{1}{2}(J_+ + J_-) \quad \text{so:}$$

$$J_x |\tfrac{3}{2}\ \tfrac{3}{2}\rangle = \tfrac{1}{2} J_- |\tfrac{3}{2}\ \tfrac{3}{2}\rangle = \tfrac{\hbar}{2}\sqrt{\tfrac{3}{2}\cdot\tfrac{5}{2} - \tfrac{3}{2}\cdot\tfrac{1}{2}}\ |\tfrac{3}{2}\ \tfrac{1}{2}\rangle = \tfrac{\sqrt{3}}{2}\hbar |\tfrac{3}{2}\ \tfrac{1}{2}\rangle$$

$$J_x |\tfrac{3}{2}\ \tfrac{1}{2}\rangle = \tfrac{1}{2} J_+ |\tfrac{3}{2}\ \tfrac{1}{2}\rangle + \tfrac{1}{2} J_- |\tfrac{3}{2}\ \tfrac{1}{2}\rangle$$

$$= \tfrac{\sqrt{3}}{2}\hbar |\tfrac{3}{2}\ \tfrac{3}{2}\rangle + \tfrac{\hbar}{2}\sqrt{\tfrac{3}{2}\cdot\tfrac{5}{2} - \tfrac{1}{2}(-\tfrac{1}{2})}\ |\tfrac{3}{2}\ -\tfrac{1}{2}\rangle = \tfrac{\sqrt{3}}{2}\hbar |\tfrac{3}{2}\ \tfrac{3}{2}\rangle + \hbar |\tfrac{3}{2}\ -\tfrac{1}{2}\rangle$$

$$J_x |\tfrac{3}{2}\ -\tfrac{1}{2}\rangle = \tfrac{1}{2} J_+ |\tfrac{3}{2}\ -\tfrac{1}{2}\rangle + \tfrac{1}{2} J_- |\tfrac{3}{2}\ -\tfrac{1}{2}\rangle$$

$$= \hbar |\tfrac{3}{2}\ \tfrac{1}{2}\rangle + \tfrac{\hbar}{2}\sqrt{\tfrac{3}{2}\cdot\tfrac{5}{2} - (-\tfrac{1}{2})(-\tfrac{3}{2})} = \hbar |\tfrac{3}{2}\ \tfrac{1}{2}\rangle + \tfrac{\sqrt{3}}{2}\hbar |\tfrac{3}{2}\ -\tfrac{3}{2}\rangle$$

So: $$\langle m = \tfrac{3}{2} | H | m = \tfrac{3}{2}\rangle = -E_0\left(\tfrac{3}{2} + \tfrac{1}{\hbar^2}\left| J_x |\tfrac{3}{2}\ \tfrac{3}{2}\rangle \right|^2\right)$$

$$= -E_0\left(\tfrac{3}{2} + \tfrac{3}{4}\right) = -\tfrac{9}{4}E_0$$

$$\langle m = -\tfrac{1}{2} | H | m = -\tfrac{1}{2}\rangle = -E_0\left(-\tfrac{1}{2} + \tfrac{1}{\hbar^2}\left| J_x |\tfrac{3}{2}\ -\tfrac{1}{2}\rangle \right|^2\right)$$

$$= -E_0\left(-\tfrac{1}{2} + 1 + \tfrac{3}{4}\right) = -\tfrac{5}{4}E_0$$

$$\langle m = -\tfrac{1}{2} | H | m = \tfrac{3}{2}\rangle = -E_0 \cdot \tfrac{1}{\hbar^2} \langle m = -\tfrac{1}{2} | J_x \circ J_x | m = \tfrac{3}{2}\rangle$$

$$= -E_0 \frac{\sqrt{3}}{2}$$

$$\langle m=\tfrac{3}{2}|H|m=-\tfrac{1}{2}\rangle = \langle m=-\tfrac{1}{2}|H|m=\tfrac{3}{2}\rangle^{*} = -E_0\frac{\sqrt{3}}{2}$$

So we have:

$$E_{ground} \leq \cos^2\theta \langle \tfrac{3}{2}|H|\tfrac{3}{2}\rangle + \sin^2\theta \langle -\tfrac{1}{2}|H|-\tfrac{1}{2}\rangle + 2\sin\theta\cos\theta \langle -\tfrac{1}{2}|H|\tfrac{3}{2}\rangle$$

$$= -E_0\left(\frac{9}{4}\cos^2\theta + \frac{5}{4}\sin^2\theta + \sqrt{3}\sin\theta\cos\theta\right) \quad (*)$$

To find the best lower bound, we want to maximize the expression in brackets. We must have that the θ derivative is zero, so:

$$-\frac{4}{2}\cos\theta\sin\theta + \frac{5}{2}\sin\theta\cos\theta + \sqrt{3}(\cos^2\theta - \sin^2\theta) = 0$$

$$\Rightarrow -\sin(2\theta) + \sqrt{3}\cos(2\theta) = 0$$

$$\Rightarrow \tan(2\theta) = \sqrt{3}$$

$$\Rightarrow 2\theta = \frac{\pi}{3} \text{ or } \frac{4\pi}{3}$$

$$\Rightarrow \theta = \frac{\pi}{6} \text{ or } \frac{2\pi}{3}$$

For $\theta = \frac{\pi}{6}$, we have: $\frac{9}{4}\cos^2\theta + \frac{5}{4}\sin^2\theta + \sqrt{3}\sin\theta\cos\theta = \frac{11}{4}$

For $\theta = \frac{2\pi}{3}$, we have: $\frac{9}{4}\cos^2\theta + \frac{5}{4}\sin^2\theta + \sqrt{3}\sin\theta\cos\theta = \frac{3}{4}$

So the best bound on the ground state energy is:

$$E_{grn} \leq -\frac{11}{4}E_0$$

OR: we can write: $(*) = -\left(\frac{7}{4} + \frac{1}{2}\cos(2\theta) + \frac{\sqrt{3}}{2}\sin(2\theta)\right)E_0$

$= -\left(\frac{7}{4} + \sin\left(2\theta + \frac{\pi}{6}\right)\right)E_0$ ← min is $-\frac{11}{4}$

PROBLEM 3 SOLUTIONS

a) We have that the spontaneous emission rate is:

$$A = \frac{\omega_0^3 |\vec{P}_{ab}|^2}{3\pi \varepsilon_0 \hbar c^3} \quad \text{where: } \omega_0 = \frac{1}{\hbar}\left(E(|210\rangle) - E(|100\rangle)\right)$$

$$\text{and } \vec{P}_{ab} = e\langle 210 | \vec{x} | 100 \rangle$$

The stimulated emission rate is

$$R = \frac{\pi}{3\varepsilon_0 \hbar^2} |\vec{P}_{ab}|^2 \cdot \rho(\omega_0) \quad \text{where } \rho(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/kT} - 1}$$

So:

$$\frac{R}{A} = \frac{\pi^2 c^3 \rho(\omega_0)}{\hbar \, \omega_0^3} = \frac{1}{e^{\hbar\omega_0/kT} - 1}$$

Here:

$$\hbar\omega_0 = (-13.6\,\text{eV}) \times \left(1.6 \times 10^{-19} \tfrac{\text{J}}{\text{eV}}\right) \cdot \left(-1 + \tfrac{1}{4}\right)$$

$$= 1.63 \times 10^{-18}\ \text{J}$$

$$kT = \left(1.38 \times 10^{-23}\ \text{J/K}\right) \times \left(5.8 \times 10^3\ \text{K}\right)$$

$$= 8.00 \times 10^{-20}\ \text{J}$$

$$\frac{R}{A} \approx \frac{1}{e^{20.4} - 1} \approx 1.4 \times 10^{-9}$$

b) To find the spontaneous transition rate, we need to calculate the matrix elements

$$\langle 210 | \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} | 100 \rangle$$

These are:

$$\int_0^{\infty} dr\, r^2 \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi\; \psi_{100} \cdot \psi_{210}^* \cdot \begin{Bmatrix} r\cos\phi \sin\theta \\ r\sin\phi \sin\theta \\ r\cos\theta \end{Bmatrix}$$

The two wavefunctions don't depend on ϕ, so the ϕ integral vanishes except for the 3rd case. We get:

$$\langle 210|z|100\rangle = \frac{1}{\sqrt{32}\,\pi a^3} \cdot \left(\int_0^{\infty} dr \frac{r^4}{a} \cdot e^{-\frac{3r}{2a}} \right) \left(\int_0^{\pi} d\theta \cos^2\theta \sin\theta \right) \cdot 2\pi$$

$$= \frac{a \cdot 2^5}{\sqrt{8} \cdot 3^5} \left(\int_0^{\infty} dx\, x^4 e^{-x} \right) \left(-\frac{1}{3} \cos^3\theta \Big|_0^{\pi} \right)$$

$$= \frac{128\sqrt{2}}{243} a$$

So: $$A = \frac{e^2 \omega_0^3 |\langle 210|z|100\rangle|^2}{3\pi\varepsilon_0 \hbar c^3} \qquad \omega_0 = |E_{100}|\left(1 - \frac{1}{4}\right)/\hbar$$

$$= \left(\frac{3}{4}\right)^3 \cdot \left(\frac{128\sqrt{2}}{243}\right)^2 \cdot a^2 \cdot \frac{\left(|E_{100}|/\hbar\right)^3 \cdot e^2}{3\pi\varepsilon_0 \hbar c^3}$$

$$= 3.98 \times 10^5\, s^{-1}$$

PROBLEM 4 SOLUTION

a) We have a two part system, with a Harmonic oscillator, and a spin half particle. For $\varepsilon = 0$, they do not interact, and the states

$$|n\rangle \otimes |S_z\rangle$$

are a basis of energy eigenstates, with:

$$E_{n,S_z} = \hbar\omega\left(n+\tfrac{1}{2}\right) - \frac{\omega}{2}\cdot\hbar S_z$$

$$= \hbar\omega\left(n + \tfrac{1}{2} - \tfrac{1}{2}S_z\right)$$

The lowest energy states are:

$|0\uparrow\rangle$: $E = \frac{1}{4}\hbar\omega$ — for later: $|1\downarrow\rangle$: $E = \frac{7}{4}\hbar\omega$

$|0\downarrow\rangle$: $E = \frac{3}{4}\hbar\omega$ — $|2\uparrow\rangle$: $E = \frac{9}{4}\hbar\omega$

$|1\uparrow\rangle$: $E = \frac{5}{4}\hbar\omega$ — $|2\downarrow\rangle$: $E = \frac{11}{4}\hbar\omega$

So $|1\uparrow\rangle$ has the 3rd lowest energy.

b) To find the state after the perturbation is taken into account, we can use that the first order shift is

$$\delta|\psi\rangle = \sum_{|\psi_m\rangle \neq |1\uparrow\rangle} \frac{\langle\psi_m|H'|1\uparrow\rangle}{E^0_{1\uparrow} - E^0_m}\cdot|\psi_m\rangle$$

use $S_-|\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2} - \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)}\,|-\tfrac{1}{2}\rangle$

Now $H'|1\uparrow\rangle = \frac{\varepsilon}{2}\left(a^\dagger S_- + aS_+\right)|1\uparrow\rangle = \frac{\varepsilon}{2}\cdot\sqrt{2}\cdot\hbar\,|2\downarrow\rangle$

So $$\delta|\psi\rangle = \frac{\varepsilon\hbar\sqrt{2}\,|2\downarrow\rangle}{2(\underbrace{E_{1\uparrow}}_{\frac{5}{4}\hbar\omega} - \underbrace{E_{2\downarrow}}_{\frac{11}{4}\hbar\omega})} = -\frac{\varepsilon}{\omega}\frac{\sqrt{2}}{3}\,|2\downarrow\rangle$$

The 2nd excited state (to 1st order in ε), is then:

$$|\psi_2\rangle = |1\uparrow\rangle - \frac{\varepsilon}{\omega}\frac{\sqrt{2}}{3}|2\downarrow\rangle$$

The probability of measuring $S_z = \frac{\hbar}{2}$ is:

$$P_\uparrow = 1 - P_\downarrow$$

$$= 1 - \frac{\varepsilon^2}{\omega^2}\cdot\frac{2}{9}$$

c) The energy of this state is unchanged at 1st order in perturbation theory, since

$$\langle 1\uparrow| a^\dagger S_- + a S_+ |1\uparrow\rangle = 0$$

To 2nd order, we get:

$$\delta E_2^{(2)} = \sum_{|\psi_n\rangle \neq |1\uparrow\rangle} \frac{|\langle\psi_n|H'|1\uparrow\rangle|^2}{E_{1\uparrow} - E_n}$$

$$= \frac{\left|\varepsilon\hbar\frac{\sqrt{2}}{2}\right|^2}{-\hbar\omega\cdot\frac{3}{2}}$$

$$= -\frac{1}{3}\frac{\varepsilon^2\hbar}{\omega}$$

So the energy to 2nd order is:

$$E_2 = \frac{5}{4}\hbar\omega - \frac{1}{3}\frac{\varepsilon^2\hbar}{\omega}$$

PROBLEM 5 SOLUTION

a) Here we have a Hamiltonian for a harmonic oscillator with frequency ω_0, perturbed by a time-dependent perturbation with:

$$H' = \frac{1}{2}\frac{k}{1+(t/t_0)^2}\cdot X^2$$

The probability of making a transition to state $|n\rangle$ at $t=\infty$ is:

$$P_n(t=\infty) = \frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty} dt\, e^{i\frac{E_n-E_0}{\hbar}t}\langle n|H'|0\rangle\right|^2$$

Here: $\langle n|H'|0\rangle = \frac{1}{2}\frac{k}{1+(t/t_0)^2}\langle n|x^2|0\rangle$

$$= \frac{1}{2}\frac{k}{1+(t/t_0)^2}\frac{\hbar}{2m\omega}\langle n|(a+a^\dagger)^2|0\rangle$$

The only state with $n>0$ for which this is nonzero is: $|2\rangle$, and:

$$\langle 2|H'|0\rangle = \frac{\hbar}{4m\omega}\frac{k}{1+(t/t_0)^2}\cdot\sqrt{2} \leftarrow \text{using } a^\dagger a^\dagger|0\rangle = a^\dagger|1\rangle = \sqrt{2}|2\rangle$$

Then:

$$P_2(t=\infty) = \frac{1}{\hbar^2}\cdot\frac{2\hbar^2 k^2}{16m^2\omega^2}\cdot\left|\int_{-\infty}^{\infty} dt\, e^{2i\omega t}\frac{1}{1+(t/t_0)^2}\right|^2$$

$$= \frac{k^2}{8m^2\omega^2}\cdot t_0^2\cdot\pi^2 e^{-4\omega t_0} \quad \left(\text{using } \int_{-\infty}^{\infty} dt\frac{e^{ikt}}{(1+t^2)} = \pi e^{-|k|}\right)$$

$$= \frac{\pi^2}{8}\frac{k^2 t_0^2}{m^2\omega^2}e^{-4\omega t_0}$$

b) We expect that the result will be reliable if this probability is much less than 1, so:

$$K \ll e^{2\omega t_0} \cdot \frac{m\omega}{\hbar_0}$$

c) The probability is zero to first order in perturbation theory, but could be non-zero at higher orders in perturbation theory. We could derive this by extending our previous derivation.

Starting from $i\hbar \frac{d}{dt}|\Psi\rangle = H|\Psi\rangle$, we let

$$|\Psi(t)\rangle = \sum c_n(t)\, e^{-iE_n^0 t/\hbar} |n\rangle$$

And $c_n(t) = c_n^0(t) + K c_n^1(t) + K^2 c_n^2(t) + \dots$

From the S.E., we get: (at order K^2)

$$\sum_n i\hbar\left(\frac{d}{dt} c_n^{(2)} e^{-iE_n t/\hbar} - \frac{iE_n}{\hbar} c_n^{(2)} e^{-iE_n t/\hbar}\right)|n\rangle$$

$$= \sum_n \hat{H}_0 c_n^{(2)} e^{-\frac{iE_n t}{\hbar}} |n\rangle + \hat{H}_1 c_n^{(1)} e^{-\frac{iE_n t}{\hbar}} |n\rangle$$

Taking the $|4\rangle$ component, we get:

$$i\hbar \frac{d}{dt} c_4^{(2)} e^{-iE_4 t/\hbar} = e^{-\frac{iE_2 t}{\hbar}} c_2^{(1)} \langle 4|\hat{H}_1|2\rangle$$

So:

$$P_4(t=\infty) \approx |c_4^{(2)}(t=\infty)|^2 = \frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty} c_2^{(1)}(t)\cdot e^{\frac{i(E_4-E_2)t}{\hbar}} \langle 4|\hat{H}_1|2\rangle\right|^2$$

We can now perform the integrals to get the result.

$$-\frac{i}{\hbar}\int_{-\infty}^{t} e^{i(E_2-E_0)t}\langle 2|H'|0\rangle$$

$$\frac{\tau_k}{1+(t/t_0)^2}\,\frac{\hbar}{2m\omega}\sqrt{12}$$