Name & Student number:

Physics 402 Exam, April 16, 2018

Multiple Choice Questions: Please write your answers in the spaces on page 4

1) A spin half particle is in an eigenstate of S_x with eigenvalue $\hbar/2$. If we measure the z component of spin for this particle,
a) we will definitely find $S_z = 0$.
b) we might find any value between $S_z = -\hbar/2$ and $S_z = \hbar/2$
c) We will find either $S_z = -\hbar/2$ or $S_z = \hbar/2$

2) A quantum system is in an eigenstate of an observable $\mathcal{O}$ with eigenvalue λ at some time $t = 0$. If we measure this observable at a later time $t = T$,
a) We will always find λ.
b) We will not always find λ.

3) Suppose that $|\Psi_1\rangle$ is some state of a particular molecule. If J_z is the operator associated with the z component of angular momentum, which of the following states states is the result of rotating the state $|\Psi_1\rangle$ about the z axis by angle $\pi/2$?
a) $\frac{\pi}{2\hbar} J_z |\Psi_1\rangle$
b) $|\Psi_1\rangle - i\frac{\pi}{2\hbar} J_z |\Psi_1\rangle$
c) $e^{-i\frac{\pi}{2\hbar} J_z} |\Psi_1\rangle$
d) $|\Psi_1\rangle + e^{-i\frac{\pi}{2\hbar} J_z} |\Psi_1\rangle$

4) If Hermitian operators $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ commute with each other, one consequence is that
a) The observables $\mathcal{A}$ and $\mathcal{B}$ are conserved.
b) There is some basis of the Hilbert space for which the basis elements are eigenstates of both $\mathcal{A}$ and $\mathcal{B}$
c) It's possible to find a basis of energy eigenstates that have definite values for $\mathcal{A}$ and $\mathcal{B}$.
d) All states have definite values for $\mathcal{A}$ and $\mathcal{B}$.

5) If $[\hat{O}, H] = 0$, where H is the Hamiltonian and $\hat{O}$ is some Hermitian operator, which of the following is generally true?
a) If $|E\rangle$ is an energy eigenstate, then $\hat{O}|E\rangle$ is an energy eigenstate with the same energy.
b) If $|E\rangle$ is an energy eigenstate, then $\hat{O}|E\rangle$ is an energy eigenstate which may have the same energy or some other energy.
c) If $|E\rangle$ is an energy eigenstate, then all other energy eigenstates may be obtained by acting on this state with $\hat{O}$ multiple times.

6) Which of the following (un-normalized) position space wavefunctions represents an eigenstate of the momentum operator for a particle in one dimension?
a) $\psi(x) = e^{ikx}$
b) $\psi(x) = \cos(kx)$
c) $\psi(x) = \delta(x - y)$
d) Both a) and b)

7) If a Hamiltonian H_0 has two degenerate states $|A\rangle$ and $|B\rangle$ and we add a perturbation H_1 to the system such that $\langle A|H_1|A\rangle = \langle B|H_1|B\rangle = 0$ we can say that
a) the energies of these states will be unchanged to first order in perturbation theory, so there will still be a degeneracy at this order.
b) the energies may be split at first order in perturbation theory; to find out, we only need to calculate $\langle A|H_1|B\rangle$.
c) the energies may be split at first order in perturbation theory; to find out, we only need to calculate $\langle E_i|H_1|A\rangle$ and $\langle E_i|H_1|B\rangle$ for all the other energy eigenstates $|E_i\rangle$.
d) the energies may be split at first order in perturbation theory; to find out, we need to calculate $\langle A|H_1|B\rangle$ and $\langle E_i|H_1|A\rangle$ and $\langle E_i|H_1|B\rangle$ for all the other energy eigenstates $|E_i\rangle$.

8) A quantum system consisting of a spin 1 particle and a spin 1/2 particle, both at fixed locations, is described by a Hilbert space of dimension
a) 2 b) 3 c) 4 d) 5 e) 6

9) For a hydrogen atom in the state $|l = 2\ s = 1/2\ J = 3/2\ M = 1/2\rangle$ in the L^2, S^2, J^2, J_z basis, if we measure S_z, the probability that we'll find $-\hbar/2$ is
a) 0
b) 1/5
c) 2/5
d) 3/5
e) 4/5
Note: there is a Clebsch-Gordon table on the formula sheet

10) Acting on a hydrogen atom state $|l\ s = 1/2\ J\ M\rangle$ in the L^2, S^2, J^2, J_z basis, the operator $\vec{L} \cdot \vec{S}$ multiplies this state by
a) $\hbar^2 l/2$
b) $\hbar^2 lM$
c) $\hbar^2(M/2 + Jl)$
d) $\frac{1}{2}\hbar^2(J(J+1) - l(l+1) - \frac{3}{4})$

11) When we estimate the energy of a quantum ground state using the variational method, the result is

a) always less than the true ground state energy, so it provides a lower bound.

b) always greater than the true ground state energy, so it provides an upper bound.

c) either greater than or less than the ground state energy, depending on the choice of trial wavefunction. By choosing different trial wavefunctions, we can establish both upper and lower bounds on the energy.

12) A quantum system is in its ground state. If we add a small time-dependent perturbation to the Hamiltonian that is nonzero during the interval $0 \leq t \leq T$, we can say that

a) the system will generally be in some linear combination of more than one of the original energy eigenstates during $0 \leq t \leq T$ but will be in the original ground state after time T.

b) the system will generally be in some linear combination of more than one of the original energy eigenstates at any time $t > 0$.

c) the system will continue to have the same energy as it did originally, but various other observables can change with time.

d) the system will always be in one of the energy eigenstates, but which state we are in can make a transition during the period $0 \leq t \leq T$.

13) The standard formulae for the stimulated emission and absorption rates in the dipole approximation assume all of the following *except*:

a) the wavelength of the radiation is long compared with the size of the molecule

b) the radiation is incoherent (i.e. without phase correlations).

c) the amplitude of the electromagnetic fields is not too large.

d) the initial and final states each have nonvanishing electric dipole moment

14) Suppose a quantum system has a number atoms with two possible states a and b, with a spontaneous emission rate A from b to a, and stimulated emission from b to a and absorption from a to b governed by a rate R. If the number of atoms at some time in states a and b are given by N_a and N_b, then the time rate of change of the number of N_a atoms will be

a) $AN_b + RN_b$

b) $AN_b - RN_a$

c) $-AN_b - RN_b + RN_a$

d) $-AN_b + RN_b - RN_a$

e) $AN_b + RN_b - RN_a$

15) In a simple quantum field theory associated with a classical system that obeys the wave equation, a particle can be understood as
a) one of the elementary constituents that make up the field through their collective motion
b) a pointlike object whose motion is what gives rise to oscillations in the field
c) a quantum of energy in a harmonic oscillator that describes a particular Fourier mode of the field
d) a standing wave in the field whose position is localized to some small region

16) The Bell inequalities show that
a) statistical results of experiments measuring the spins of entangled particles cannot be explained by a model in which measurement outcomes are predetermined and the measurements do not affect each other.
b) the non-deterministic behaviour of quantum mechanics cannot really be true: there must be some underlying description in which all states have definite values for physical observables.
c) quantum entanglement can be used to transmit information over large distances faster than is possible according to the rules of special relativity.
d) the bells of St. Clement's are always at least as loud as, but may be louder than the bells of St. Martin's.

Multiple Choice Answers (1 point each):

1	2	3	4	5	
6	7	8	9	10	
11	12	13	14	15	16

Long answer (6 points each): Please write your answers in an exam booklet

Problem 1

For a hydrogen atom in a $|100\rangle$ state, the state of the proton and electron spins at $t = 0$ is

$$|\psi(t=0)\rangle = |\uparrow\rangle \otimes |\downarrow\rangle = |m_p = \frac{1}{2}, m_e = -\frac{1}{2}\rangle \ .$$

The Hamiltonian for this system can be approximated by

$$H = \frac{E}{\hbar^2} J^2$$

where $\vec{J} = \vec{J_e} + \vec{J_p}$. If we measure the z component of the electron spin after time T, what is the probability that we will find $S_z = \hbar/2$?

Problem 2

Suppose we would like to estimate the ground state energy for a single spin $J = 3/2$ particle with Hamiltonian

$$H = -E_0(\frac{1}{\hbar}J_z + \frac{1}{\hbar^2}J_x^2) \qquad E_0 > 0 \tag{1}$$

using the variational method with a trial state

$$\cos(\theta)|m = \frac{3}{2}\rangle + sin(\theta)|m = -\frac{1}{2}\rangle$$

where θ is a variational parameter. What is the smallest upper bound on the ground state energy that can be obtained with a trial state of this form?

Problem 3

The temperature at the surface of the sun is about $5.8 \times 10^3 K$.

a) At this temperature, what is the ratio R_{stim}/R_{spon} between the rates for stimulated and spontaneous emissions from the $|210\rangle$ state of hydrogen to the ground state?

b) What is the spontaneous transition rate? *These may be useful:*

$$\psi_{100}(\vec{x}) = \frac{1}{\sqrt{\pi} a^{\frac{3}{2}}} e^{-\frac{r}{a}} \qquad \psi_{210}(\vec{x}) = \frac{1}{\sqrt{32\pi} a^{\frac{3}{2}}} \frac{r}{a} e^{-\frac{r}{2a}} \cos\theta \qquad (x, y, z) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r\cos\theta)$$

$$\int_0^\infty ds s^3 e^{-s} = 6 \qquad \int_0^\pi d\theta \sin\theta \cos^2\theta = \frac{2}{3} \qquad \frac{(E_{100}/\hbar)^3 e^2}{3\pi\epsilon_0 \hbar c^3} = 6.08 \times 10^{26} m^{-2} s^{-1}$$

Problem 4

A particle in a 1D harmonic oscillator potential interacts with another nearby spin half particle whose location is fixed. The system may be described by a Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - \frac{\omega}{2}S_z + \frac{\epsilon}{2}(a^\dagger S_- + aS_+)$$

where ϵ can be treated as a small parameter and $S_\pm = S_x \pm iS_y$ as usual.

a) For $\epsilon = 0$, what is the second excited state (i.e. the state with the 3rd lowest energy)?

b) Now suppose ϵ is positive but $\epsilon \ll \omega$. If the system is in the second excited state (after taking into account the perturbation), and we measure S_z for the spin, what is the probability that we will find $\hbar/2$ (in the approximation of first order perturbation theory)?

c) What is the energy of this state, including the leading non-vanishing dependence on ϵ?

Problem 5

Consider a quantum system described by a Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}k(t)x^2 \ ,$$

where

$$k(t) = m\omega_0^2 + \frac{K}{1 + \left(\frac{t}{t_0}\right)^2} \ .$$

a) At $t = -\infty$, the system starts in the ground state of the Hamiltonian at that time. For small values of the parameter K, estimate the probability that the system will have made a transition to some other state.

b) For which values of K do you expect your result to be reliable?

c) Is it possible that the system at $t = \infty$ will be found in the state $|4\rangle$? Explain. If yes, can you derive an expression for the probability of this or explain how such an expression might be derived?

The following may or may not be useful:

$$\int_{-\infty}^{\infty} dt \frac{e^{ikt}}{(1+t^2)} = \pi e^{-|k|} \qquad \int_{-\infty}^{\infty} dt \frac{e^{ikt}}{(1+t^2)^2} = 4\pi e^{-|k|}$$