**15)** The quantum description of the electromagnetic field is mathematically equivalent to
a) the quantum theory of a collection of harmonic oscillators.
b) the quantum mechanics of many particles interacting via a Coulomb potential.
c) the quantum mechanics of many particles interacting only by exchange forces.
d) None of the above: the electromagnetic field is an inherently classical field that interacts with quantum systems such as atoms and molecules.

**16)** In order to give a position-space description of the state of a quantum system with two particles in one dimension, we can use
a) two wavefunctions $\psi_1(x)$ and $\psi_2(x)$, one for each particle.
b) a single wavefunction $\psi(x_1, x_2)$ depending on two position variables.
c) two wavefunctions, $\psi_1(x_1, x_2)$ and $\psi_2(x_1, x_2)$ each depending on two variables.
d) Either a) or b).

**Multiple Choice Answers:**

1 c 2 b 3 c 4 c 5 a

6 d 7 a 8 a 9 a 10 b

11 c 12 c 13 b 14 d 15 a

16 b

**Problem 1**

Consider the state $|\Psi\rangle = |n=3, l=2, m=1\rangle \otimes |s_z = \frac{1}{2}\rangle$ of an electron in a hydrogen atom.

a) For this state, if we measure $J^2$, what values might we obtain and what are the corresponding probabilities? (3 points)

b) If we perform an infinitesimal rotation of this state around the $x$ axis by angle $\theta$, what is the change $\delta|\Psi\rangle$ in the state? Write your answer in the $|n\ l\ m\rangle \otimes |s_z\rangle$ basis. (*Note: don't worry about whether the rotation is clockwise or counterclockwise; just give the answer up to an overall sign*) (3 points)

a) Using the CG table, we have:

For $\ell = 2 \times s = \frac{1}{2}$:

$$|n=3\ \ell=2\ m=1\rangle \otimes |s_z = \tfrac{1}{2}\rangle = \frac{2}{\sqrt{5}}|n=3\ \ell=2\ J=\tfrac{5}{2}\ M=\tfrac{3}{2}\rangle - \frac{1}{\sqrt{5}}|n=3\ \ell=2\ J=\tfrac{3}{2}\ M=\tfrac{3}{2}\rangle$$

The allowed $J^2$ values are thus $\hbar^2 \cdot J(J+1) = \begin{cases} \frac{35}{4}\hbar^2 & \text{probability } \frac{4}{5} \\ \frac{15}{4}\hbar^2 & \text{probability } \frac{1}{5} \end{cases}$

b) The infinitesimal rotation generator is the angular momentum operator $J_x = L_x + S_x$. The change in the state is

$$\delta|\Psi\rangle = \pm\frac{i}{\hbar}\theta\left(L_x|\Psi\rangle + S_x|\Psi\rangle\right)$$

Using $L_x = \frac{1}{2}(L_+ + L_-)$ and $S_x = \frac{1}{2}(S_+ + S_-)$, we get:

$$\delta|\Psi\rangle = \pm\frac{i}{\hbar}\theta\Big(\tfrac{1}{2}\cdot\hbar\sqrt{2\cdot3 - 1\cdot2}\,|n=3\ \ell=2\ m=2\rangle\otimes|s_z=\tfrac{1}{2}\rangle$$
$$+\tfrac{1}{2}\cdot\hbar\sqrt{2\cdot3-1\cdot0}\,|n=3\ \ell=2\ m=0\rangle\otimes|s_z=\tfrac{1}{2}\rangle$$
$$+\frac{\hbar}{2}|n=3\ \ell=2\ m=1\rangle\otimes|s_z=-\tfrac{1}{2}\rangle\Big)$$
$$= \pm\frac{i}{2}\theta\left(2|3\ 2\ 2\rangle\otimes|\tfrac{1}{2}\rangle + \sqrt{6}|3\ 2\ 0\rangle\otimes|\tfrac{1}{2}\rangle + |3\ 2\ 1\rangle\otimes|-\tfrac{1}{2}\rangle\right)$$

**Problem 2**

Abigail would like to estimate the ground state energy of a particle of mass $m$ in a potential $V(x) = \lambda x^4$. She decides to use the variational method with a family of trial states $|\Psi(\omega)\rangle = |0\rangle_\omega$, i.e. the ground state of a harmonic oscillator with frequency $\omega$ (which she allows to vary). Determine the best lower bound on the energy that can be obtained using this method. (6 points)

*Hint: you can avoid calculating integrals by using the fact that the Hamiltonian for this system can be written as*

$$H = \frac{p^2}{2m} + \lambda x^4 = (\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2) + (\lambda x^4 - \frac{1}{2}m\omega^2 x^2)\ ,$$

*where the first bracketed term is the harmonic oscillator Hamiltonian for frequency $\omega$.*

We have: $E_0 \leq \langle \Psi(\omega)|H|\Psi(\omega)\rangle$

$$= \langle \Psi(\omega)|H_\omega|\Psi(\omega)\rangle + \lambda\langle\Psi(\omega)|x^4|\Psi(\omega)\rangle - \frac{1}{2}m\omega^2\langle\Psi(\omega)|x^2|\Psi(\omega)\rangle$$

$$= \frac{\hbar\omega}{2} + \lambda\left(\frac{\hbar}{2m\omega}\right)^2\langle\Psi(\omega)|(a+a^\dagger)^4|\Psi(\omega)\rangle - \frac{1}{2}m\omega^2\left(\frac{\hbar}{2m\omega}\right)\cdot\langle\Psi(\omega)|\cancel{x^2}|\Psi(\omega)\rangle$$

$(a+a^\dagger)^2$

PROBABLY A BIT EASIER:

Alternatively just write $x^4$ and $p^2$ using creation + annihilation operators and calculate $\frac{1}{2m}\langle 0_\omega|p^2|0_\omega\rangle + \lambda\langle 0_\omega|x^4|0_\omega\rangle$

We have: $(a+a^\dagger)|\Psi_\omega\rangle = (a+a^\dagger)|0\rangle_\omega = |1\rangle_\omega$

so $\langle\Psi_\omega|(a+a^\dagger)^2|\Psi_\omega\rangle = \langle 1_\omega|1_\omega\rangle = 1$

$(a+a^\dagger)^2|\Psi_\omega\rangle = (a+a^\dagger)|1\rangle_\omega = \sqrt{2}|2_\omega\rangle + |0\rangle_\omega$

so $\langle\Psi(\omega)|(a+a^\dagger)^4|\Psi(\omega)\rangle = (\sqrt{2}\langle 2| + \langle 0|)(\sqrt{2}|2\rangle + |0\rangle) = 3$

Then $E_0 \leq \frac{\hbar\omega}{2} + \lambda\left(\frac{\hbar}{2m\omega}\right)^2\cdot 3 - \frac{\hbar\omega}{4} = \frac{\hbar}{4}\omega + \left(\frac{3\lambda\hbar^2}{4m^2}\right)\frac{1}{\omega^2}$

Minimizing over $\omega$, we find the min at $\frac{\hbar}{4} - \frac{3\lambda\hbar^2}{2m^2}\cdot\frac{1}{\omega^3} = 0$, so

$\omega_{min} = \sqrt[3]{\frac{6\lambda\hbar}{m^2}}$, So the best bound is

$$E_0 \leq \frac{3\sqrt[3]{6}}{8}\frac{\lambda^{1/3}\hbar^{4/3}}{m^{2/3}}$$

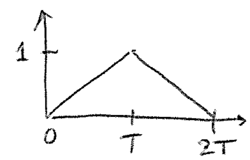
**Problem 3**

A particle of mass $m$ is in the ground state of an infinite square well potential of width $a$. Starting at $t = 0$, the potential in the left half of the well increases at a constant rate from 0 to $V$ in time $T$ and then decreases back to zero at a constant rate in time $T$. If $V$ is small, what is the probability that the particle will be found in the first excited state of the well at time $2T$? *Hint: the formula sheet should help.* (6 points)

We have a time dependent perturbation

$$H'(t) = \begin{cases} V f(t) & 0 \leq x \leq \frac{a}{2} \\ 0 & \text{elsewhere} \end{cases}$$

where $f(t)$ looks like:




The transition probability is

$$P_{1\to 2} = \left| \frac{1}{\hbar} \int_0^{2T} dt\, H'_{21}(t)\, e^{i\omega_0 t} \right|^2 \quad \text{where} \quad \omega_0 = \frac{E_2 - E_1}{\hbar} = \frac{3\pi^2 \hbar}{2ma^2}$$

Here

$$\begin{aligned} H'_{21}(t) &= \langle 2 | H'(t) | 1 \rangle \\ &= V \int_0^{\frac{a}{2}} \psi_2^*(x) \psi_1(x)\, dx\, f(t) \\ &= V f(t) \cdot \int_0^{\frac{a}{2}} \frac{2}{a} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{L}\right) = \frac{2V}{a} f(t) \int_0^{\frac{a}{2}} 2 \sin^2\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi x}{a}\right) dx \\ &= \frac{4V}{a} f(t)\, \frac{1}{3} \sin^3\left(\frac{\pi x}{a}\right) \cdot \frac{a}{\pi} \Big|_0^{a/2} \\ &= \frac{4V}{3\pi} f(t) \end{aligned}$$

Then

$$P_{1\to 2} = \frac{16 V^2}{9\pi^2 \hbar^2} \left| \int_0^{2T} dt\, f(t)\, e^{i\omega t} \right|^2 = \boxed{\frac{256 V^2}{9\pi^2 \hbar^2 \omega^4 T^2} \sin^4\left(\frac{\omega T}{2}\right)}$$

where we used

$$\int_0^{2T} dt\, f(t)\, e^{i\omega t} = \int_0^T dt\, \frac{t}{T} e^{i\omega t} + \int_T^{2T} \frac{(2T - t)}{T} e^{i\omega t} = \int_0^T dt\, \frac{t}{T} \left( e^{i\omega t} + e^{i\omega(2T - t)} \right)$$

$$= -\frac{T}{\omega^2 T^2} \left(1 - e^{i\omega T}\right)^2 = -\frac{e^{i\omega T}}{\omega^2 T} \left(2i \sin\left(\frac{\omega T}{2}\right)\right)^2$$

**Problem 4**

A cavity contains $3N$ molecules with two available states. There are $2N$ molecules in the ground state $|0\rangle$ with energy $E_0$ and $N$ molecules in an excited state $|1\rangle$ with energy $E_1 > E_0$. The cavity contains incoherent electromagnetic radiation; the energy density per unit frequency is described by some function $\rho(\omega)$. (6 points)

a) Describe the various physical processes that could cause the number of molecules in each state to change with time.

b) If the matrix elements for the components of the electric dipole operator are given by

$$\langle 1|\mathcal{P}_x|0\rangle = \langle 1|\mathcal{P}_y|0\rangle = \langle 1|\mathcal{P}_z|0\rangle = p\,,$$

what condition on $\rho(\omega)$ ensures that the number of molecules in each state will remain constant on average?

a) We can have $|0\rangle \to |1\rangle$ through absorption of radiation, $|1\rangle \to |0\rangle$ through stimulated emission, and $|1\rangle \to |0\rangle$ through spontaneous emission.

b) The rate (per atom) for the absorption and stimulated emission processes are:

$$R = \frac{\pi}{3\varepsilon_0 \hbar^2} \vec{\mathcal{P}}_{ab} \cdot \vec{\mathcal{P}}_{ab}^{\,*} \rho(\omega_0) \quad \text{where} \quad \omega_0 = \frac{E_1 - E_0}{\hbar}$$

$$= \frac{\pi p^2}{\varepsilon_0 \hbar^2} \rho(\omega_0)$$

The rate for spontaneous emission is $A = \dfrac{\omega_0^3 |\vec{\mathcal{P}}_{ab}|^2}{3\pi\varepsilon_0 \hbar c^3} = \dfrac{\omega_0^3 p^2}{\pi \varepsilon_0 \hbar c^3}$

The time rate of change of molecules in the $|0\rangle$ state is:

$$\frac{dN_0}{dt} = -N_0 \cdot R + N_1 \cdot R + N_1 \cdot A$$

We want this to be zero for $N_0 = 2N$ and $N_1 = N$, so:

$$0 = -2N \cdot R + N \cdot R + N \cdot A$$

$$\Rightarrow \underline{R = A}$$

$$\Rightarrow \boxed{\rho(\omega_0) = \frac{\hbar}{\pi^2} \frac{\omega_0^3}{c^3}}$$

## Problem 5

Consider two nearby spin half particles at fixed location. The particles sit in a magnetic field which leads to a term

$$H_1 = \frac{C}{\hbar}((S_1)_z + \frac{1}{2}(S_2)_z) \tag{1}$$

in the Hamiltonian; they also have an interaction between their magnetic moments that results in a spin-spin interaction

$$H_2 = \frac{A}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2 = +\frac{A}{2\hbar^2}(S_{tot}^2 - S_1^2 - S_2^2) \tag{2}$$

Here, $\vec{S}_1$ and $\vec{S}_2$ are the angular momentum operators for the two spins and $\vec{S}_{tot} = \vec{S}_1 + \vec{S}_2$

a) Assuming that $A$ and $C$ are positive and $A << C$ what is the energy of the ground state to the first nonzero order in $A$? What is the ground state in the limit that $A \to 0$? (3 points)

b) Now suppose that $C << A$. In this case, what is the energy of the ground state to the first nonzero order in $C$? What is the ground state in the limit that $C \to 0$? (3 points)

c) Make a qualitative graph of all energy levels of the system as $A$ is varied from large negative values to large positive values for fixed $C$ (i.e. plot $E$ vs $A$ for each energy eigenvalue, all on the same graph). (2 points) $C > 0$

a) For $A=0$, the ground state is $|\downarrow\downarrow\rangle$ with energy $\frac{C}{\hbar}\left(-\frac{\hbar}{2} - \frac{\hbar}{4}\right) = -\frac{3}{4}C$

We can find the energy to order $A$ using non-degenerate perturbation theory. Since $|\downarrow\downarrow\rangle$ is the state $|J=1\ M=-1\rangle$ we have:

$$\delta E = \langle\psi_0|H_2|\psi_0\rangle = \frac{A}{2}\left(J(J+1) - S_1(S_1+1) - S_2(S_2+1)\right) = \frac{A}{2}\cdot\left(2 - \frac{3}{4} - \frac{3}{4}\right) = \frac{A}{4}$$

So the energy to order $A$ is $-\frac{3}{4}C + \frac{1}{4}A + \mathcal{O}(A^2)$ ← actually, 0

b) For $C \to 0$ the Hamiltonian is diagonal in the $J^2$, $J_z$ basis, and the lowest energy is for the $J=0$ state, with $E_0 = \frac{A}{2\hbar^2}\left(0 - \frac{3\hbar^2}{4} - \frac{3\hbar^2}{4}\right) = -\frac{3}{4}A$

The ground state is $|0\ 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$.

To find the energy at order $C$, we can again use non-deg. perturbation theory.

We have: $\langle 00|H_1|00\rangle = \left(\frac{1}{\sqrt{2}}\langle\uparrow\downarrow| - \frac{1}{\sqrt{2}}\langle\downarrow\uparrow|\right)\frac{C}{\hbar}\cdot\frac{\hbar}{2}\left(\frac{1}{2}|\uparrow\downarrow\rangle + \frac{1}{2}|\downarrow\uparrow\rangle\right)\frac{1}{\sqrt{2}} = 0.$

← this is $|J=1\ M=0\rangle$ × constant.

So we need to go to second order. We get

$$\delta E_2 = \sum_{n\neq 0}\frac{|\langle E_n|H_1|00\rangle|^2}{E_0 - E_n} = \frac{|\langle 1\,0|H_1|00\rangle|^2}{E^0_{|00\rangle} - E^0_{|10\rangle}} = \frac{\left|\left(\frac{1}{\sqrt{2}}\langle\uparrow\downarrow| + \frac{1}{\sqrt{2}}\langle\downarrow\uparrow|\right)\frac{C}{4}\left(\frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle\right)\right|^2}{-\frac{3}{4}A - \frac{1}{4}A}$$

$$= -\frac{C^2}{16A}$$

So the energy is $-\frac{3}{4}A - \frac{C^2}{16A} + \mathcal{O}(C^3)$

e) For this part, we will just find the exact eigenvalues.

First, the states $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ are eigenstates of both $H_1$ and $H_2$, with eigenvalues $\frac{3}{4}C + \frac{A}{4}$ and $-\frac{3}{4}C + \frac{A}{4}$ respectively.

For the remaining states, $|00\rangle$ and $|10\rangle$, the matrix elements for $H_1 + H_2$ are:

$$\begin{pmatrix} \langle 00|H_1+H_2|00\rangle & \langle 00|H_1+H_2|10\rangle \\ \langle 10|H_1+H_2|00\rangle & \langle 10|H_1+H_2|10\rangle \end{pmatrix} = \begin{pmatrix} -\frac{3A}{4} & \frac{C}{4} \\ \frac{C}{4} & \frac{A}{4} \end{pmatrix}$$

The eigenvalues are solutions of $\lambda^2 + \frac{A}{2}\lambda - \frac{3A^2}{16} - \frac{C^2}{16}$. So

$$\lambda = -\frac{A}{4} \pm \frac{1}{4}\sqrt{4A^2 + C^2}$$

For $\lambda = -\frac{A}{4} + \frac{1}{4}\sqrt{4A^2 + C^2}$ we get $\frac{C}{4}$ for $A=0$

$\frac{A}{4} + \frac{C^2}{16A}$ for large +ve $A$

$-\frac{3A}{4} - \frac{C^2}{16A}$ for large -ve $A$

For $\lambda = -\frac{A}{4} - \frac{1}{4}\sqrt{4A^2 + C^2}$ we get $-\frac{C}{4}$ for $A=0$

$-\frac{3A}{4} - \frac{C^2}{16A}$ for large +ve $A$

$+\frac{A}{4} + \frac{C^2}{16A}$ for large -ve $A$

Putting everything together, we get: