① a) For any hamiltonian H, and any state $|\psi\rangle$, the ground state energy is less than or equal to $\langle\psi|H|\psi\rangle$.

b) We have allowed transitions when $\Delta l = \pm 1$ and $\Delta m = \pm 1, 0$, so the possible endpoints are

$$|311\rangle, \; |310\rangle, \; |211\rangle \text{ and } |210\rangle$$

c) We have $\text{spin } \tfrac{1}{2} \times \text{spin } \tfrac{1}{2} \times \text{ang. mom } 1$

$$\rightarrow (\text{spin } 0 + \text{spin } 1) \times \text{ang mom } 1$$

$$\rightarrow j = 1 + (j=0 + j=1 + j=2)$$

$\therefore$ $j=0 \rightarrow 1$ state

$2 \times j = 1 \rightarrow 6$ states

$j = 2 \rightarrow 5$ states.

2 a) The ground state has a spin ↑ and spin ↓ electron in the $|100\rangle$ state, so:

$$E_g = 2 \times Z^2 \times E_0 = 18 E_0$$

b) The state must be antisymmetric under exchange of the two electrons, so we have

$$|\psi_g\rangle = \frac{1}{\sqrt{2}}\left(|1\,0\,0\,\tfrac{1}{2}\rangle \otimes |1\,0\,0\,-\tfrac{1}{2}\rangle - |1\,0\,0\,-\tfrac{1}{2}\rangle \otimes |1\,0\,0\,\tfrac{1}{2}\rangle\right)$$

c) There is no orbital angular momentum, and the antisymmetric combination of spin $\tfrac{1}{2}$ states corresponds to total angular momentum 0, so we will obtain $J_z = 0$

d) For the 1st excited level, there will be an electron in either $|1\,0\,0\,\tfrac{1}{2}\rangle$ or $|1\,0\,0\,-\tfrac{1}{2}\rangle$ and a second electron in $|2\,0\,0\,\pm\tfrac{1}{2}\rangle$ or $|2\,1\,m\,\pm\tfrac{1}{2}\rangle$ for a total of $2 \times 8 = 16$ states with energy $Z^2(E_0 + E_0/4) = \frac{45}{4} E_0$.

③ a) We have: $P_{0\to n} = |c_n|^2$ where

$$c_n = -\frac{i}{\hbar}\int_0^T e^{i\omega_0 t'} H'_{n0}(t')\,dt'$$

Here $\omega_0 = \frac{E_n - E_0}{\hbar} = n\omega$ and

$$H'_{n0} = \langle n | \tfrac{1}{2} m \Delta(t) x^2 | 0 \rangle$$

$$= \tfrac{1}{2} m \Delta(t) \frac{\hbar}{2m\omega} \langle n | (a+a^\dagger)^2 | 0 \rangle$$

$$= \tfrac{1}{2} m \Delta(t) \frac{\hbar}{2m\omega} \langle n | (a+a^\dagger) | 1 \rangle$$

$$= \frac{\sqrt{2}}{2} m \Delta(t) \frac{\hbar}{2m\omega} \delta_{n,2} \qquad \text{(assuming } n>0\text{)}$$

$\therefore$ c_n and $P_{0\to n}$ are 0 for $n=1$, $n \geq 3$ (at 1st order) and

$$c_2 = -\frac{i}{\hbar}\int_0^T e^{2i\omega t} \cdot \frac{\sqrt{2}}{4} \cdot \frac{\hbar}{\omega} \cdot \Delta \sin\left(\frac{t}{T}\pi\right) dt$$

$$= -\frac{\sqrt{2}\,i}{4\omega}\Delta \int_0^T e^{2i\omega t} \frac{1}{2i}\left(e^{i\frac{t}{T}\pi} - e^{-i\frac{t}{T}\pi}\right)$$

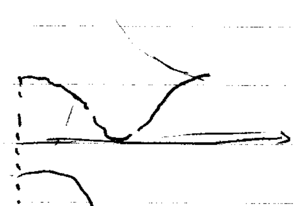
$$= -\frac{\sqrt{2}\Delta}{8\omega}\left(\frac{1}{i(2\omega+\frac{\pi}{T})}\left(-e^{2i\omega T} - 1\right) - \frac{1}{i(2\omega - \frac{\pi}{T})}\left(-e^{2i\omega T} - 1\right)\right)$$

$$= \frac{\sqrt{2}}{8 i\omega}\Delta \cdot \left(e^{2i\omega T} + 1\right)\left\{\frac{-2\pi/T}{4\omega^2 - \pi^2/T^2}\right\}$$

$$= i\,\frac{\sqrt{2}\pi}{4}\,\frac{\Delta}{\omega T}\,\frac{\left(e^{2i\omega T}+1\right)}{4\omega^2 - \pi^2/T^2}$$

$$\therefore P_{0\to 2} = \frac{\pi^2 \Delta^2}{8\omega^2 T^2}\,\frac{4\cos^2(\omega T)}{|4\omega^2 - \pi^2/T^2|^2}$$

$$= \frac{\pi^2}{32}\left(\frac{\Delta T}{\omega}\right)^2 \frac{\cos^2(\omega T)}{|(\omega T)^2 - (\frac{\pi}{2})^2|^2}$$



b) $P_{0\to 2}$ should be small, so we should have $\Delta \ll \omega/T$

6 a) We have ground state energy $\frac{3}{2}\hbar\omega$ for the state $|0\rangle$ such that $a_x|0\rangle = a_y|0\rangle = a_z|0\rangle = 0$.
The first excited level, with energy $\frac{5}{2}\hbar\omega$, has 3 states $a_x^\dagger|0\rangle$, $a_y^\dagger|0\rangle$, $a_z^\dagger|0\rangle$.

b) We have $L_z = xp_y - yp_x$

$$= \sqrt{\frac{\hbar}{2m\omega}} \times \sqrt{\frac{m\hbar\omega}{2}}\Big[(a_x+a_x^\dagger)(i)(a_y^\dagger-a_y) - (a_y+a_y^\dagger)(i)(a_x^\dagger-a_x)\Big]$$

$$= \hbar i(a_x a_y^\dagger - a_y a_x^\dagger)$$

To find the eigenstates of L_z in the first excited level, we can demand

$$L_z\left(c_x a_x^\dagger|0\rangle + c_y a_y^\dagger|0\rangle + c_z a_z^\dagger|0\rangle\right) = \lambda\left(c_i a_i^\dagger|0\rangle\right)$$

$$\Rightarrow \hbar i\left(c_x a_y^\dagger|0\rangle - c_y a_x^\dagger|0\rangle\right) = \lambda\left(c_x a_x^\dagger|0\rangle + c_y a_y^\dagger|0\rangle + c_z a_z^\dagger|0\rangle\right)$$

$$\Rightarrow \hbar i c_x = \lambda c_y$$
$$-\hbar i c_y = \lambda c_x$$
$$0 = \lambda c_z$$

$\therefore$ ~~The solutions are eigenvalues~~

We can have $\lambda = 0$ for $\vec{c} = (0,0,1)$

$\lambda = \hbar$ for $\vec{c} = (\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 0)$

$\lambda = -\hbar$ for $\vec{c} = (\frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}, 0)$

$\therefore$ $L_z = 0$: $a_z^\dagger|0\rangle$

$L_z = \hbar$: $\frac{1}{\sqrt{2}}(a_x^\dagger + i a_y^\dagger)|0\rangle$

$L_z = -\hbar$: $+\frac{1}{\sqrt{2}}(a_x^\dagger - i a_y^\dagger)|0\rangle$

c) These states must have total angular momentum $l=1$, since this is the only value with 3 indep states of L_z. Thus, the states in b) are eigenstates distinguished by their eigenvalues of L^2 & L_z. In this basis, H_1 is diagonal, so

$$\Delta E = \alpha \hbar^2 \cdot 2 + \beta \hbar m \quad \text{where } m = \pm 1, 0.$$

d) The nth excited level has states.

$$(a_x^\dagger)^{n_x} (a_y^\dagger)^{n_y} (a_z^\dagger)^{n_z} |0\rangle$$

where $n_x + n_y + n_z = n$. Equivalently, we can write a basis

$$|n_+, n_-, n_0\rangle = (a_x^\dagger + i a_y^\dagger)^{n_+} (a_x^\dagger - i a_y^\dagger)^{n_-} {a_z^\dagger}^{n_0} |0\rangle$$

Now, using $[L_z, a_z^\dagger] = 0$, $[L_z, a_x^\dagger + i a_y^\dagger] = \hbar(a_x^\dagger + i a_y^\dagger)$, $[L_z, a_x^\dagger - i a_y^\dagger] = -\hbar(a_x^\dagger - i a_y^\dagger)$, we have:

$$L_z |n_+ \, n_- \, n_0\rangle = \hbar (n_+ - n_-) |n_+ \, n_- \, n_0\rangle$$

The maximum value of L_z with $n_+ + n_- + n_0$ is $\hbar n$, for $(a_x^\dagger + i a_y^\dagger)^n |0\rangle$. There ~~is one state~~ must then be one set of states with $l = n$.

For $L_z = \hbar(n-1)$ we have only one state, and this must be part of the $l=n$ ~~set~~ subspace

For $L_z = \hbar(n-2)$, we have 2 indep. states, so there must also be a $l = n-2$ set of states.

Continuing in this way, we find that the nth excited level contains states with $l = n, n-2, n-4, \ldots$. For each set, m can be $-l, -l+1, \ldots l-1, l$