1 reading for lecture 3

We've seen that each physical observable in a quantum system can be associated with an orthonormal basis of states with particular values for that observable. Next lecture, we're going to recall that same information (an orthonormal basis plus a real number associated with each basis element) is what we use to describe a *Hermitian operator*. Please read through the following (you don't have to follow everything) and think about the following reading question:

Question For a two dimensional real vector space, what are some examples of linear maps from the vector space to itself? Can you describe these both mathematically and in geometrical language? If you can, come to class with a couple of specific examples in mind.

1.1 Operators

Given a Hilbert space, a *linear operator* or simply *operator* is defined as a linear map from the vector space to itself, i.e. a map

$$|v\rangle \to \hat{\mathcal{O}}|v\rangle \tag{1}$$

satisfying

$$\hat{\mathcal{O}}(z_1|v_1\rangle + z_2|v_2\rangle) = z_1\hat{\mathcal{O}}|v_1\rangle + z_2\hat{\mathcal{O}}|v_2\rangle \ . \tag{2}$$

The set of operators has the mathematical structure of an *algebra*. This means that they form a complex vector space (i.e. we can add operators and multiply them by a constant), with the additional structure of being able to multiply operators. This multiplication is defined by

$$(\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2)|v\rangle = \hat{\mathcal{O}}_1(\hat{\mathcal{O}}_2|v\rangle) \ . \tag{3}$$

The algebra of operators includes an "identity" element, the operator **1** that takes every vector to itself.

Matrix representation of an operator

By the linearity property, the action of an operator is completely specified by its action on any basis of vectors: if $|v\rangle = \sum c_n|e_n\rangle$ then $\hat{\mathcal{O}}|v\rangle = \sum c_n\hat{\mathcal{O}}|e_n\rangle$.

Acting on any basis vector, the operator must give some linear combination of basis vectors. We can express this as

$$\hat{\mathcal{O}}|e_n\rangle \equiv \sum_m \mathcal{O}_{mn}|e_m\rangle \ , \tag{4}$$

where

$$\mathcal{O}_{mn} \equiv \langle e_m|\mathcal{O}|e_n\rangle \ . \tag{5}$$

are called the matrix elements of the operator $\hat{\mathcal{O}}$.

The matrix $\mathcal{O}_{mn}$ gives a representation of the operator $\hat{\mathcal{O}}$ in a particular basis. If a vector $|v\rangle$ is represented in the same basis by coefficients c_n, the vector $\mathcal{O}|v\rangle$ is represented by coefficients c'_m given by[1]

$$c'_m = \sum_n \mathcal{O}_{mn} c_n \ ; \tag{6}$$

That is, we write the coefficients c_n as a column vector and multiply by the matrix O_{mn} to get the new coefficients.

It often helps to think of these linear operators as performing some type of geometrical transformation on the vector space. We can have various qualitatively different types including operators that stretch things in various ways, operators that preserve lengths similar to rotations, or operators that project onto some lower dimensional subspace.

Eigenvectors and eigenvalues

The action of an operator may be particularly simple on certain vectors. We say that $|v\rangle$ is an *eigenvector* of $\hat{\mathcal{O}}$ with *eigenvalue* λ if

$$\hat{\mathcal{O}}|v\rangle = \lambda|v\rangle \ . \tag{7}$$

Sometimes there can be different eigenvectors with the same eigenvalue. In this case, any linear combination of these eigenvectors is an eigenvector with this eigenvalue.

For some operators, it is possible to choose a basis of vectors which are all eigenvectors of $\hat{\mathcal{O}}$. In this case, $\hat{\mathcal{O}}$ is represented simply as a diagonal matrix.

Hermitian operators

A particular example of this type with special physical relevance in quantum mechanics is the Hermitian operator. For these operators, it is possible to choose the basis of eigenvectors to be orthonormal, and all of the eigenvalues are real. We can visualize the action of a Hermitian operator as stretching the space along each eigenvector direction by a multiplicative factor given by the eigenvalue.

For a Hermitian operator, if we denote the eigenvalues by λ_n and the eigenvectors by $|\lambda_n\rangle$, the matrix elements in this basis are simply

$$\mathcal{O}_{mn} = \lambda_m \delta_{mn} \ , \tag{8}$$

[1] To see this, we note that $c'_m = \langle e_m|\mathcal{O}|v\rangle = \langle e_m|\mathcal{O}(\sum_n c_n|e_n\rangle) = \sum_n \mathcal{O}_{mn}c_n$.

i.e. a diagonal matrix with the eigenvalues along the diagonal.

In another basis, a Hermitian operator will generally not be diagonal. But we can recognize it as Hermitian since in an orthonormal basis, the *adjoint* of the corresponding matrix (defined as the complex conjugate of the transpose) must be equal to the matrix itself:

$$\mathcal{O}_{nm} = \mathcal{O}^*_{mn} \; . \tag{9}$$

We can express this property in a basis independent language: for an operator $\hat{\mathcal{O}}$, we define the adjoint operator $\hat{\mathcal{O}}^\dagger$ to be the operator satisfying

$$\langle \chi | \hat{\mathcal{O}}^\dagger | \psi \rangle = \langle \psi | \hat{\mathcal{O}} | \chi \rangle^* \; . \tag{10}$$

With this definition, a Hermitian operator is an operator satisfying $\hat{\mathcal{O}}^\dagger = \hat{\mathcal{O}}$. A basic theorem in linear algebra states that this is true if and only if there exists an orthonormal basis of eigenvectors with real eigenvalues.

Properties of the Adjoint

From the definition of the adjoint, we can immediately see the following properties that will be useful below

- $(z\hat{\mathcal{O}})^\dagger = z^* \hat{\mathcal{O}}^\dagger$
- If $|w\rangle = \hat{\mathcal{O}}|v\rangle$, then $\langle w|u\rangle = \langle v|\hat{\mathcal{O}}^\dagger|u\rangle$
- $(\hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2)^\dagger = \hat{\mathcal{O}}_2^\dagger \hat{\mathcal{O}}_1^\dagger$
- If $\hat{\mathcal{O}}|v\rangle = \lambda|v\rangle$, then $\langle v|\mathcal{O}^\dagger|w\rangle = \lambda^*\langle v|w\rangle$

From the third property here, we see that the product of two Hermitian operators is generally not Hermitian. However, it follows from the properties of the adjoint that the combination $i[\hat{\mathcal{A}}, \hat{\mathcal{B}}]$ is Hermitian if $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ are Hermitian.