PROBLEM SET 8 SOLUTIONS

WEBWORK: We have: $J_x = \frac{1}{2}(J_+ + J_-)$. Denoting the J_z eigenstates by $\underset{-1,0,1}{|M\rangle}$ with $J_z|M\rangle = \hbar M|M\rangle$, we have:

$$J_\pm |M\rangle = \hbar\sqrt{J(J+1) - M(M\pm 1)}\,|M\pm 1\rangle \quad \text{with } J=1$$

So:

$$J_x|1\rangle = \frac{1}{2}(J_+|1\rangle + J_-|1\rangle) = \frac{1}{2}J_-|1\rangle = \frac{\sqrt{2}}{2}\hbar|0\rangle$$

$$J_x|0\rangle = \frac{1}{2}(J_+|0\rangle + J_-|0\rangle) = \frac{\sqrt{2}}{2}\hbar|1\rangle + \frac{\sqrt{2}}{2}\hbar|-1\rangle$$

$$J_x|-1\rangle = \frac{1}{2}(J_+|-1\rangle + J_-|-1\rangle) = \hbar\frac{\sqrt{2}}{2}|0\rangle$$

So the matrix elements of J_x in the J_z basis are:

$$\hbar\begin{pmatrix} 0 & \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & 0 \end{pmatrix}$$

③ We'll work in the J_z basis. For $H_0 = QJ_z^2$, we see that the J_z eigenstates are already eigenstates of H_0:

$$H_0|M\rangle = QJ_z^2|M\rangle = Q\hbar^2M^2|M\rangle$$

So we have state $|0\rangle$ with energy 0 and states $|1\rangle$ and $|-1\rangle$ with energy $Q\hbar^2$.

Now we add a perturbation

$$H_1 = \lambda(J_x^2 - J_y^2) = \lambda\left(\left(\frac{J_+ + J_-}{2}\right)^2 - \left(\frac{J_+ - J_-}{2i}\right)^2\right)$$

$$= \frac{1}{2}\lambda\left(J_+^2 + J_-^2\right)$$

The first order shift for $|0\rangle$ is:

$$\langle 0|H_1|0\rangle = \frac{1}{2}\lambda\langle 0|J_+^2 + J_-^2|0\rangle$$

$$= 0$$

The other states are degenerate, so we need degenerate perturbation theory to find the first order energy shifts. We compute:

$$\begin{pmatrix} \langle 1|H_1|1\rangle & \langle 1|H_1|-1\rangle \\ \langle -1|H_1|1\rangle & \langle -1|H_1|-1\rangle \end{pmatrix} = \hbar^2\lambda\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(e.g. $\langle 1|H_1|-1\rangle = \frac{1}{2}\lambda\hbar^2 \cdot \langle 1|J_+^2|-1\rangle = \lambda\hbar^2$

The eigenvectors of this matrix are $\frac{1}{\sqrt{2}}\binom{1}{1}$ with eigenvalue $\hbar^2\lambda$ and $\frac{1}{\sqrt{2}}\binom{1}{-1}$ with eigenvalue $-\hbar^2\lambda$.
So the state $\frac{1}{\sqrt{2}}(|1\rangle + |-1\rangle)$ has energy $Q\hbar^2 + \lambda\hbar^2$ to first order and the state $\frac{1}{\sqrt{2}}(|1\rangle - |-1\rangle)$ has energy $Q\hbar^2 - \lambda\hbar^2$ to first order. These are actually the exact energies, since we can check that they are exact eigenvectors of $H_0 + \lambda H_1$ with these same eigenvalues. (Generally, we would do this by writing down the 3×3 matrix for $H_0 + \lambda H_1$ and finding the eigenvalues and eigenvectors)

(4) We have $|\Psi\rangle = |n=2, l=1, m=1\rangle \otimes |S_z = \frac{1}{2}\rangle$

If we rotate around the x axis by an infinitesimal angle ε, the change in the state is:

$$\delta|\Psi\rangle = -\frac{i\varepsilon}{\hbar}\hat{J}_x|\Psi\rangle$$

$$= -\frac{i\varepsilon}{\hbar}(\hat{L}_x + \hat{S}_x)|\Psi\rangle$$

$$= -\frac{i\varepsilon}{\hbar}\left(\frac{1}{2}\left(\hat{L}_+ + \hat{L}_-\right) + \hat{S}_x\right)|2\ 1\ 1\rangle \otimes |\tfrac{1}{2}\rangle$$

$$= -\frac{i\varepsilon}{\hbar}\cdot\left(\frac{1}{2}\hat{L}_+|2\ 1\ 1\rangle \otimes |\tfrac{1}{2}\rangle + \frac{1}{2}\hat{L}_-|2\ 1\ 1\rangle \otimes |\tfrac{1}{2}\rangle + |2\ 1\ 1\rangle \otimes \hat{S}_x|\tfrac{1}{2}\rangle\right)$$

$$= -\frac{i\varepsilon}{\hbar}\left(\hbar\frac{\sqrt{2}}{2}|2\ 1\ 0\rangle \otimes |\tfrac{1}{2}\rangle + |2\ 1\ 1\rangle \otimes \frac{\hbar}{2}|-\tfrac{1}{2}\rangle\right)$$

$$= -\frac{i\varepsilon}{2}\left(\sqrt{2}|2\ 1\ 0\rangle \otimes |\tfrac{1}{2}\rangle + |2\ 1\ 1\rangle \otimes |-\tfrac{1}{2}\rangle\right)$$

used that S_x is represented as $\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the S_z basis

EXTRA: to rotate by $\frac{\pi}{6}$, we need to apply $e^{-\frac{i}{\hbar}\frac{\pi}{6}\hat{J}_x}$. We first note that $|\Psi\rangle$ is the state $|n=2, l=1, J=\frac{3}{2}, M=\frac{3}{2}\rangle$. We can solve our problem using the same method that we use to do time evolution: write our initial state in terms of J_x eigenstates and then act with $e^{-\frac{i}{\hbar}\frac{\pi}{6}J_x}$.

First, lets find the matrix for $J_x = \frac{1}{2}(J_+ + J_-)$

Since $J_+ \,|\, J=\frac{3}{2}, M\rangle = \hbar\sqrt{J(J+1) - M(M+1)}\,|J=\frac{3}{2}\; M+1\rangle$

and $J_- \,|J=\frac{3}{2}, M\rangle = \hbar\sqrt{J(J+1) - M(M-1)}\,|J=\frac{3}{2}\; M-1\rangle$

we see that:

$$J_+|M=\tfrac{3}{2}\rangle = 0 \qquad J_+|M=\tfrac{1}{2}\rangle = \sqrt{3}\hbar|M=\tfrac{3}{2}\rangle$$

$$J_+|M=-\tfrac{1}{2}\rangle = 2\hbar|M=\tfrac{1}{2}\rangle \qquad J_+|-\tfrac{3}{2}\rangle = \sqrt{3}\hbar|M=-\tfrac{1}{2}\rangle$$

so $$J_+ = \hbar\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_- = J_+^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$J_x = \frac{\hbar}{2}\begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}.$$

Now we can find the eigenvectors,

using that the eigenvalues are $-\frac{3}{2}\hbar, -\frac{1}{2}\hbar,$ $\frac{1}{2}\hbar, \frac{3}{2}\hbar$:

e.g. $\lambda = \frac{3}{2}\hbar$:
$$\begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \frac{3}{2} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$\Rightarrow \frac{\sqrt{3}}{2} b = \frac{3}{2} a \quad \frac{\sqrt{3}}{2} a + c = \frac{3}{2} b \quad b + \frac{\sqrt{3}}{2} d = \frac{3}{2} c$$
$$\frac{\sqrt{3}}{2} c = \frac{3}{2} d$$

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \frac{1}{\sqrt{8}} \begin{pmatrix} 1 \\ \sqrt{3} \\ \sqrt{3} \\ 1 \end{pmatrix}$$ where we have normalized.

We can similarly find the other three eigenvectors

Now, our initial state is $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ so we have:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = c_1 \vec{V}_{\frac{3}{2}\hbar} + c_2 \vec{V}_{\frac{1}{2}\hbar} + c_3 \vec{V}_{-\frac{1}{2}\hbar} + c_4 \vec{V}_{\frac{3}{2}\hbar}$$

e.g $C_1 = \vec{V}_{\frac{3}{2}\hbar} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{8}}$.

Finally, we have at time t:

$$C_1 \vec{V}_{\frac{3}{2}\hbar} \cdot e^{-\frac{i}{\hbar}\cdot\frac{\pi}{6}\cdot\frac{3}{2}\hbar} + C_2 \vec{V}_{\frac{1}{2}\hbar} e^{-\frac{i}{\hbar}\frac{\pi}{6}\cdot\frac{1}{2}\hbar}$$

$$+ c_3 \vec{V}_{-\frac{1}{2}\hbar} e^{\frac{i}{\hbar}\frac{\pi}{6}\frac{1}{2}\hbar} + c_4 \vec{V}_{-\frac{3}{2}\hbar} e^{\frac{i}{\hbar}\frac{\pi}{6}\frac{3}{2}\hbar}$$