① a) We have states $|n_1\ n_2\rangle$ with energies

$$E = E_1 + E_2 = \hbar\omega(n_1 + \tfrac{1}{2}) + \hbar\omega(n_2 + \tfrac{1}{2})$$
$$= \hbar\omega(n_1 + n_2 + 1)$$

The possible energies are thus $\hbar\omega$, $2\hbar\omega$, $3\hbar\omega$, and we have N states for energy $N\hbar\omega$:

$$(N-1, 0),\ (N-2, 1),\ \ldots,\ (1, N-2),\ (0, N-1)$$

b) We now add $\lambda H_1 = \lambda xy$

$$= \lambda \frac{\hbar}{2m\omega}(a_x + a_x^\dagger)(a_y + a_y^\dagger)$$

The ground state is $|0\ 0\rangle$. This is not degenerate, so we have $\delta E = \lambda\langle 0\ 0|H_1|0\ 0\rangle$

$$= \lambda\frac{\hbar}{2m\omega}\langle 0\ 0|(a_x + a_x^\dagger)(a_y + a_y^\dagger)|0\ 0\rangle$$
$$= 0$$

The next states are $|0\ 1\rangle$ and $|1\ 0\rangle$ with energy $2\hbar\omega$. These are degenerate, so we need to calculate the matrix

$$\begin{pmatrix} \langle 01|H_1|01\rangle & \langle 01|H_1|10\rangle \\ \langle 10|H_1|01\rangle & \langle 10|H_1|10\rangle \end{pmatrix}$$

and find its eigenvalues and eigenvectors.

We have:

$$H_1 |0\ 1\rangle = \frac{\hbar}{2m\omega}(a_x + a_x^\dagger)(a_y + a_y^\dagger)|0\ 1\rangle$$
$$= \frac{\hbar}{2m\omega}\left(|1\ 0\rangle + \sqrt{2}|1\ 2\rangle\right)$$

$$H_1 |1\ 0\rangle = \frac{\hbar}{2m\omega}(a_x^\dagger + a_x)(a_y + a_y^\dagger)|1\ 0\rangle$$
$$= \frac{\hbar}{2m\omega}\left(|0\ 1\rangle + \sqrt{2}|2\ 1\rangle\right)$$

So our matrix is

$$\begin{pmatrix} \langle 01|H_1|01\rangle & \langle 01|H_1|10\rangle \\ \langle 10|H_1|01\rangle & \langle 10|H_1|10\rangle \end{pmatrix} = \frac{\hbar}{2m\omega}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This has eigenvalues $+\frac{\hbar}{2m\omega}$ and $-\frac{\hbar}{2m\omega}$ with eigenstates $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ respectively. So the state $\frac{1}{\sqrt{2}}\left(|0\ 1\rangle + |1\ 0\rangle\right)$ gets an energy shift $\delta E = \lambda \frac{\hbar}{2m\omega}$ and the state $\frac{1}{\sqrt{2}}\left(|0\ 1\rangle - |1\ 0\rangle\right)$ gets a shift $-\lambda \cdot \frac{\hbar}{2m\omega}$.

3 a) The spatial and spin parts of the Hamiltonian are independent, so we can take a basis of eigenstates to be

$$|n_\uparrow\rangle = |n\rangle \otimes |S_z = \tfrac{\hbar}{2}\rangle \quad \text{and} \quad |n_\downarrow\rangle = |n\rangle \otimes |S_z = -\tfrac{\hbar}{2}\rangle$$

We have: $$H|n_\uparrow\rangle = \hbar\omega(n+\tfrac{1}{2}) + \frac{\hbar\omega}{2} = \hbar\omega(n+1)$$

$$H|n_\downarrow\rangle = \hbar\omega(n+\tfrac{1}{2}) - \frac{\hbar\omega}{2} = \hbar\omega n$$

$\therefore$ We have: energy $0 \rightarrow$ no degeneracy $|0_\downarrow\rangle$

energy $\hbar\omega n$ $(n>1)$: degeneracy 2 $|n_\downarrow\rangle, |n-1_\uparrow\rangle$

b) We now add $H' = \alpha \cdot x S_x$

$$= \alpha \cdot \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)\cdot\frac{1}{2}(S_+ + S_-)$$

The shift in the ground state energy in 1st order P.T. is

$$\Delta E = \langle 0_\downarrow|\alpha\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)\frac{1}{2}(S_+ + S_-)|0_\downarrow\rangle = 0.$$

c) We need degenerate P.T. here.

$$H'|0_\uparrow\rangle = \frac{\alpha}{2}\sqrt{\frac{\hbar}{2m\omega}}\, a^\dagger S_- |0_\uparrow\rangle$$
$$= \frac{\alpha}{2}\sqrt{\frac{\hbar}{2m\omega}}\,|1_\downarrow\rangle\cdot\hbar$$

$$H'|1_\downarrow\rangle = \frac{\alpha}{2}\sqrt{\frac{\hbar}{2m\omega}}\,(S_+a + S_+a^\dagger)\,|1_\downarrow\rangle$$
$$= \frac{\alpha}{2}\sqrt{\frac{\hbar}{2m\omega}}\cdot\hbar\,(|0_\uparrow\rangle + \sqrt{2}|2_\uparrow\rangle)$$

In the degenerate subspace, H' has matrix elements

$$\langle 0\uparrow|H'|0\uparrow\rangle = 0 \qquad \langle 0\uparrow|H'|1\downarrow\rangle = \frac{\alpha}{2}\hbar\sqrt{\frac{\hbar}{2m\omega}}$$

$$\langle 1\downarrow|H'|0\uparrow\rangle = \frac{\alpha}{2}\sqrt{\frac{\hbar}{2m\omega}}\,\hbar \qquad \langle 1\downarrow|H'|1\downarrow\rangle = 0$$

This matrix, $\frac{\alpha}{2}\hbar\sqrt{\frac{\hbar}{2m\omega}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, has eigenvalues $\pm\frac{\alpha}{2}\hbar\sqrt{\frac{\hbar}{2m\omega}}$, so the degeneracy is split and we end up with:

$$\frac{1}{\sqrt{2}}(|0_\uparrow\rangle + |1\downarrow\rangle) \rightarrow \text{Energy } \hbar\omega + \frac{\alpha}{2}\hbar\sqrt{\frac{\hbar}{2m\omega}} + \ldots$$

$$\frac{1}{\sqrt{2}}(|0\uparrow\rangle - |1\downarrow\rangle) \rightarrow \text{Energy } \hbar\omega - \frac{\alpha}{2}\hbar\sqrt{\frac{\hbar}{2m\omega}} + \ldots$$