PROBLEM SET 6 SOLUTIONS

WEBWORK:

1.

(4 points) **setPS6/prob1.pg**

Consider the first excited state $|1\rangle$ of a harmonic oscillator with mass m and frequency ω. If we add a perturbation $H' = \lambda x^2$ to the Hamiltonian, the first order shift in the energy of the state (i.e. the correction at order λ) is $\lambda\hbar/(m\omega)$ times

________ .

The shift is $\langle 1|H'|1\rangle$

$$= \lambda \langle 1|x^2|1\rangle$$

$$= \lambda \frac{\hbar}{2m\omega} \langle 1|(a+a^\dagger)^2|1\rangle$$

We have: $(a+a^\dagger)|1\rangle = |0\rangle + \sqrt{2}|2\rangle$ so

$$\langle 1|(a+a^\dagger)(a+a^\dagger)|1\rangle = \left(\langle 0| + \sqrt{2}\langle 2|\right)\left(|0\rangle + \sqrt{2}|2\rangle\right)$$

$$= 1 + 2 = 3$$

Then $\Delta E = \lambda \frac{\hbar}{m\omega} \cdot \frac{3}{2}$.

(2)

(4 points) **local/setPS6/Problem1.pg**

Consider the third excited state $|3\rangle$ of a harmonic oscillator. If we add a perturbation $H' = \lambda x^3$, the first order shift in the energy of the state (i.e. the correction at order λ) is ______ .

The state shifts to

$|3\rangle + \lambda|\delta\psi\rangle + \ldots$

where $|\delta\psi\rangle$ can be written as a superposition of energy eigenstates of the original harmonic oscillator:

$|\delta\psi\rangle \; = \sum_n c_n|n\rangle$

For how many n is the coefficient c_n nonzero? ______ .

We have $\Delta E = \langle 3|H'|3\rangle$

$$= \lambda\langle 3|x^3|3\rangle$$

$$= \lambda\langle 3|\left(\sqrt{\frac{\hbar}{2m\omega}}\right)^3(a+a^\dagger)^3|3\rangle$$

But $(a+a^\dagger)^3|3\rangle$ is a linear combination of $|6\rangle, |4\rangle, |2\rangle$, and $|0\rangle$, so the inner product with $\langle 3|$ vanishes and $\Delta E = 0$.

For the state shift, we have:

$$c_n = \frac{\langle n|H'|3\rangle}{E_3 - E_n}$$

From the previous part, this will be nonzero for $n = 0, 2, 4, 6$, so four states contribute.

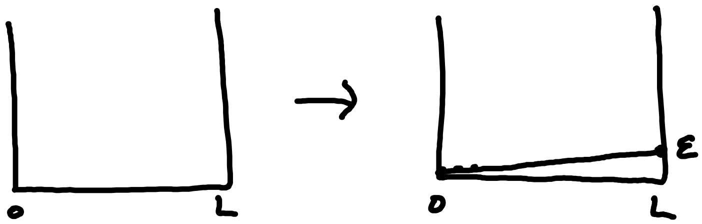
③

(4 points) **setPS6/problem2.pg**

The energy eigenstates for a particle in an infinite square well potential between $x=0$ and $x=L$ have wavefunctions

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right).$$

Consider the n =37 energy eigenstate. If we now modify the potential slightly by making it increase linearly from 0 at $x=0$ to some potential energy ϵ at $x=L$, the change in energy of this state to leading order in ϵ will be ϵ times



For the n-th eigenstate, we have

$$\Delta E = \langle n | H' | n \rangle$$

Here $H' = \tilde{V}(\hat{x})$ where $\tilde{V}(x)$ is a function

$$\tilde{V}(x) = \begin{cases} \varepsilon x/L & 0 \leq x \leq L \\ 0 & \text{elsewhere} \end{cases}$$

Then $\langle n | H' | n \rangle = \int_0^L dx\, \psi_n^*(x)\, \tilde{V}(x)\, \psi_n(x)$

$$= \frac{2}{L}\int_0^L dx \sin^2\left(\frac{n\pi x}{L}\right) \cdot \varepsilon x/L$$

set $x = y \cdot L$ $\quad = 2\varepsilon \int_0^1 dy \sin^2(n\pi y) \cdot y$

$$= \frac{1}{2}\varepsilon$$

2a) We need to compute

$$\Delta E_1 = \langle 1 | H_1 | 1 \rangle$$

$$= \frac{1}{100} \frac{\omega}{\hbar} \langle 1 | p^2 x^2 | 1 \rangle$$

We can write $p = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$ $x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$

so $p^2 x^2 = -\frac{\hbar^2}{4}(a - a^\dagger)^2 (a + a^\dagger)^2$

So $\Delta E_1 = -\frac{\hbar\omega}{400} \cdot \langle 1 | (a - a^\dagger)^2 (a + a^\dagger)^2 | 1 \rangle$

Using $a|n\rangle = \sqrt{n}\,|n-1\rangle$, $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$, we get:

$$(a + a^\dagger)^2 |1\rangle = (a + a^\dagger)(|0\rangle + \sqrt{2}|2\rangle) = 3|1\rangle + \sqrt{6}|3\rangle$$

$$(a - a^\dagger)^2 |1\rangle = (a - a^\dagger)(|0\rangle - \sqrt{2}|2\rangle) = -3|1\rangle + \sqrt{6}|3\rangle$$

$\langle 1 | (a - a^\dagger)^2 (a + a^\dagger)^2 | 1 \rangle$ is the inner product between these two so equals $-9 + 6 = -3$.

We conclude that the first order energy shift is

$$\Delta E_1 = \frac{3}{400}\hbar\omega$$

b) The correction to the state is:

$$\sum_{n \neq 1} |n\rangle \frac{\langle n | H_1 | 1 \rangle}{E_1 - E_n}$$

We have: $H_1 |1\rangle = -\frac{\hbar\omega}{400}(a - a^\dagger)^2 (a + a^\dagger)^2 |1\rangle$

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$$-\frac{\hbar\omega}{400}(a-a^\dagger)^2\left[3|1\rangle+\sqrt{6}|3\rangle\right]$$

$$=-\frac{\hbar\omega}{400}(a-a^\dagger)\left[3|0\rangle-2\sqrt{6}|4\rangle\right]$$

$$=-\frac{\hbar\omega}{400}\left[-3|1\rangle-4\sqrt{6}|3\rangle+2\sqrt{30}|5\rangle\right]$$

So $\quad \langle 3|H_1|1\rangle = \frac{\sqrt{6}}{100}\hbar\omega$

$$\langle 5|H_1|1\rangle = -\frac{\sqrt{30}}{200}\hbar\omega$$

and all other matrix elements in our sum vanish.

Thus: $\delta|\psi_1\rangle = |3\rangle\cdot\dfrac{\langle 3|H_1|1\rangle}{\underbrace{E_1-E_3}_{2\hbar\omega}} + |5\rangle\cdot\dfrac{\langle 5|H_1|1\rangle}{\underbrace{E_1-E_5}_{4\hbar\omega}}$

$$=\frac{\sqrt{6}}{200}\cdot|3\rangle-\frac{\sqrt{30}}{800}\cdot|5\rangle$$

Adding a perturbation $\lambda H_1 = \lambda p^3$, the first order shift in the ground state energy vanishes, since $\langle 0|p^3|0\rangle = 0$. The second order shift is:

$$\delta E_2 = \lambda^2 \sum_{n\neq 0} \frac{|\langle n|p^3|0\rangle|^2}{E_0 - E_n}$$

$$= \lambda^2 \left(\sqrt{\frac{\hbar\omega m}{2}}\right)^6 \sum_{n\neq 0} \frac{|\langle n|(a-a^\dagger)^3|0\rangle|^2}{E_0 - E_n}$$

$$(a-a^\dagger)^3|0\rangle = -(a-a^\dagger)^2|1\rangle$$

$$= (a-a^\dagger)\left(-|0\rangle + \sqrt{2}|2\rangle\right)$$

$$= 3|1\rangle - \sqrt{6}|3\rangle$$

So $\langle 1|(a-a^\dagger)^3|0\rangle = 3$ and $\langle 3|(a-a^\dagger)^3|0\rangle = -\sqrt{6}$

Then $$\delta E_2 = \lambda^2 \cdot \left(\frac{\hbar\omega m}{2}\right)^3 \left\{\frac{9}{(-\hbar\omega)} + \frac{6}{(-3\hbar\omega)}\right\}$$

$$= -\frac{11}{8}\lambda^2 \cdot \hbar^2\omega^2 \cdot m^3$$