#1. Let's consider the equation

$$x^5 + \varepsilon x - 1 = 0 \qquad (*)$$

For $\varepsilon = 0$, we have $x^5 - 1 = 0$ so $x = 1$. Now for small ε, we can expand the solution as:

$$x_*(\varepsilon) = 1 + \varepsilon \cdot c_1 + \varepsilon^2 \cdot c_2 + \dots$$

Plugging in to $(*)$ and keeping terms up to order ε, we have:

$$(1 + \varepsilon \cdot c_1 + \dots)^5 + \varepsilon(1 + \varepsilon \cdot c_1 + \dots) - 1 = 0$$

$$\Rightarrow 1 + 5\varepsilon \cdot c_1 + \dots + \varepsilon - 1 + \dots = 0$$

$$\Rightarrow 0 + \varepsilon(5c_1 + 1) + \mathcal{O}(\varepsilon^2) = 0$$

Since this is supposed to vanish for any ε, all powers of ε must separately vanish, so:

$$5c_1 + 1 = 0 \rightarrow c_1 = -\frac{1}{5}$$

The solution to order ε is

$$x_*(\varepsilon) = 1 - \frac{1}{5}\varepsilon + \dots$$

For $\varepsilon = \frac{1}{100}$, we get:

$$x_* \approx 1 - \frac{1}{500} = 0.998$$

#2 : See class notes or Griffiths

③ a) For this question, we want to make use of the key results that $a|n\rangle = \sqrt{n}\,|n-1\rangle$, $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$. We have:

$$(a^\dagger)^n|0\rangle = (a^\dagger)^{n-1}|1\rangle = (a^\dagger)^{n-2}\sqrt{2}\,|n\rangle = \ldots = \sqrt{n!}\,|n\rangle$$

So: $\langle 0|a^n a^{\dagger n}|0\rangle$ = inner product between $(a^\dagger)^n|n\rangle$ and itself $= \left(\sqrt{n!}\,\langle n|\right)\left(\sqrt{n!}\,|n\rangle\right) = n!$

Alernatively, we can just use $a|0\rangle = 0$ and:

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1.$$

So:

$$\begin{aligned}\langle 0|aa^\dagger|0\rangle &= \langle 0|a^\dagger a + 1|0\rangle \\ &= \langle 0|a^\dagger a|0\rangle + \langle 0|0\rangle \\ &= 1\end{aligned}$$

For the other ones, we can use:

$$\begin{aligned}a(a^\dagger)^n &= aa^\dagger a^\dagger \cdots a^\dagger \\ &= aa^\dagger a^\dagger \cdots a^\dagger - a^\dagger a a^\dagger \cdots a^\dagger \\ &\quad + a^\dagger a a^\dagger \cdots a^\dagger - a^\dagger a^\dagger a a^\dagger \cdots a^\dagger \\ &\quad + \ldots \\ &\quad + a^\dagger \cdots a^\dagger a a^\dagger - a^\dagger \cdots a^\dagger a \\ &\quad + a^\dagger \cdots a^\dagger a \\ &= [a, a^\dagger]\, a^\dagger \cdots a^\dagger \\ &\quad + a^\dagger [a, a^\dagger]\, a^\dagger \cdots a^\dagger \\ &\quad + \ldots + a^\dagger \cdots a^\dagger [a, a^\dagger] \\ &\quad + a^\dagger \cdots a^\dagger a \\ &- n(a^\dagger)^{n-1} + (a^\dagger)^n a\end{aligned}$$

So $$\langle 0|a^2 a^{\dagger 2}|0\rangle = \langle 0|a[2a^\dagger + a^{\dagger 2}a]|0\rangle$$
$$= 2\langle 0|a a^\dagger|0\rangle$$
$$= 2$$

$$\langle 0|a^3 a^{\dagger 3}|0\rangle = \langle 0|a^2[3a^{\dagger 2} + a^{\dagger 3}a]|0\rangle$$
$$= 3\langle 0|a^2 a^{\dagger 2}|0\rangle$$
$$= 6$$

b) We have:

$$\langle 2|x^2|2\rangle = \langle 2|\left[\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)\right]^2|2\rangle$$
$$= \frac{\hbar}{2m\omega}\langle 2|(a+a^\dagger)(a+a^\dagger)|2\rangle$$

used that $\langle 2|a^2|2\rangle = \langle 2|a^{\dagger 2}|2\rangle = 0$

$$= \frac{\hbar}{2m\omega}\langle 2|aa^\dagger + a^\dagger a|2\rangle$$
$$= \frac{\hbar}{2m\omega}\langle 2|\sqrt{3}\cdot\sqrt{3} + \sqrt{2}\cdot\sqrt{2}|2\rangle$$
$$= \frac{5\hbar}{2m\omega}$$

EXTRA #1 For a Hermitian operator $\hat{O}$, the associated infinitesimal transformation is:

$$\hat{\tau}(\varepsilon) = (1 - i\hat{O}\varepsilon + \dots)$$

and the family of transformations obtained by doing this many times is

$$\hat{\tau}(a) = e^{-i\hat{O}a}$$

For $\hat{O} = \hat{X}$, we get $\hat{\tau}_x(a) = e^{-i\hat{X}a}$. If $|\Psi\rangle$ has wavefn $\langle x|\Psi\rangle = \psi(x)$, then: $\hat{\tau}_x(a)|\Psi\rangle$ has wavefn. $\langle x| e^{-i\hat{X}a}|\Psi\rangle$ → can act to the left, since $= \langle\Psi|(e^{-i\hat{X}a})^\dagger|x\rangle^*$

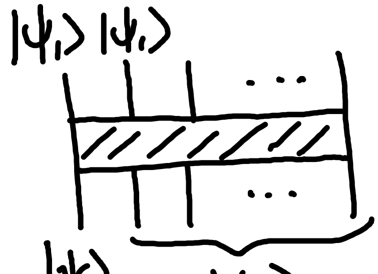
$$= \langle x|e^{-ixa}|\Psi\rangle \qquad = \langle\Psi|e^{i\hat{X}a}|x\rangle^*$$

$$= e^{-ixa}\langle x|\Psi\rangle \qquad = \langle\Psi|e^{ixa}|x\rangle^*$$

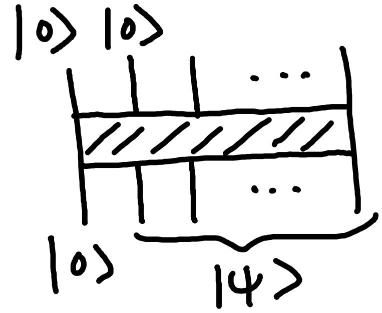
$$= e^{-ixa}\psi(x) \qquad = \langle x|e^{-ixa}|\Psi\rangle$$

For this wavefunction, the norm squared is the same as before, so all position measurements will have the same probabilities for their outcomes. But momentum measurements will be different. In fact, by the almost symmetrical relation between position + momentum in QM, we might guess that $\hat{\tau}_x(a)$ is a "translation" in momentum, taking $|p\rangle$ to $|p-a\rangle$. For example, we can check that if $\langle\psi|\hat{p}|\psi\rangle = \bar{p}$ then $\langle\psi|\tau_x^\dagger(a)\,\hat{p}\,\tau_x(a)|\psi\rangle = \bar{p} - \hbar a$.

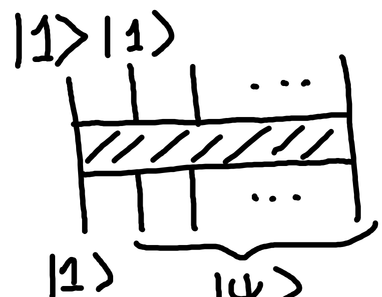
EXTRA #2: We will consider the more general case. Suppose it were possible to come up with a quantum circuit that can duplicate a qubit state as shown.



Then if we send in $|0\rangle$ in the first bit, the output for the first two bits will be $|0\rangle|0\rangle$.



If we send in $|1\rangle$, the output for the first two bits will be $|1\rangle|1\rangle$.



But since the circuit just performs a linear operation, if we send in

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$

the output for the first two bits will be a superposition of the previous outputs: $\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$. On the other hand, what we wanted was:

$$|\psi\rangle|\psi\rangle = \frac{1}{2}(|0\rangle+|1\rangle)(|0\rangle+|1\rangle)$$
$$= \frac{1}{2}(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle)$$

So we can't duplicate an arbitrary input state.

EXTRA 3:

We have: $\hat{P}|p\rangle = p|p\rangle$ ①

and: $|\Psi\rangle = \int dp\ \chi(p)\,|p\rangle$ ②

where: $\chi(p) = \langle p|\Psi\rangle$ ③

We want to find $\psi_p(x) = \langle x|p\rangle$. From ①, we have: $\langle x|\hat{P}|p\rangle = p\langle x|p\rangle$

$$\Rightarrow \frac{\hbar}{i}\frac{d}{dx}\psi_p(x) = p\,\psi_p(x)$$

$$\Rightarrow \psi_p(x) = A\cdot e^{\frac{ipx}{\hbar}}$$

We want to normalize so that $\langle p|q\rangle = \delta(p-q)$,

So: $$\int dx\,\psi_p^*(x)\psi_q(x) = \delta(p-q)$$

$$\Rightarrow A^2\int dx\ e^{i\frac{(q-p)\cdot x}{\hbar}} = \delta(p-q)$$

$$\Rightarrow A^2\cdot h\cdot\frac{1}{2\pi}\int ds\ e^{i(q-p)\cdot s} = \delta(p-q)$$

$$\Rightarrow A^2\cdot h\cdot\ \delta(p-q) = \delta(p-q) \quad \text{so } A = \frac{1}{\sqrt{h}}$$

Finally: $P(p>0) = \int_0^\infty dp\,|\chi(p)|^2$ where $\chi(p) = \frac{1}{\sqrt{h}}\int dx\,e^{-ipx/\hbar}\phi(x)$

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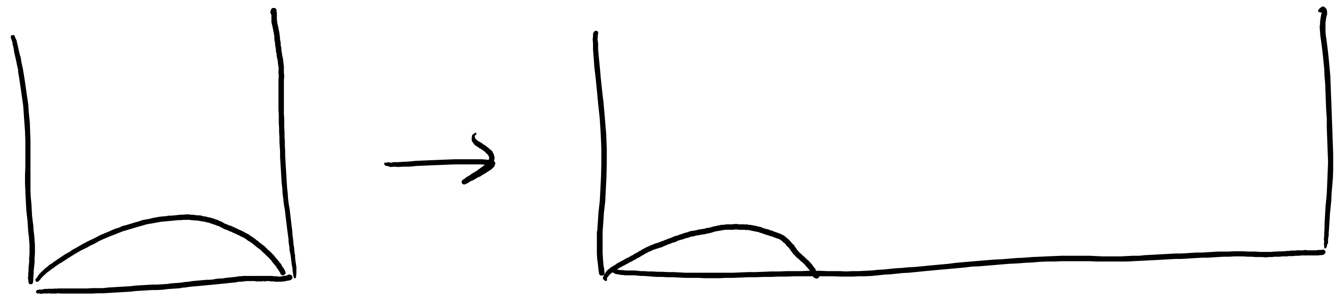

After the change in potential, the allowed energies are $E_n = \dfrac{n^2 \pi^2 \hbar^2}{2m(3L)^2}$. To find the corresponding probabilities, we take:

$$P_n = |\langle E_n | \psi \rangle|^2$$

Using: $\langle \chi | \psi \rangle = \int dx \, \chi^*(x) \cdot \psi(x)$, we get

$$P_n = \left| \int_0^{3L} \psi_n^*(x) \psi(x) \, dx \right|^2$$

where $\psi(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L} x\right) & 0 \leq x \leq L \\ 0 & \text{elsewhere} \end{cases}$

is our original wavefunction, and

$$\psi_n(x) = \sqrt{\frac{2}{3L}} \sin\left(\frac{n \pi x}{3L}\right)$$

are the energy eigenstate wavefunctions for the new well.

Working out the integrals, we get:

$$P_n = \frac{4}{3L^2}\left| \int_0^L \sin\left(\frac{\pi x}{L}\right) \cdot \sin\left(\frac{n\pi x}{3L}\right) dx \right|^2$$

$$= \frac{4}{3}\left| \int_0^1 \sin(\pi x) \cdot \sin\left(\frac{n\pi x}{3}\right) dx \right|^2$$

$$=^{*} \begin{cases} \frac{1}{3} & n = 3 \\ \frac{4}{3\pi^2} \sin^2\left(\frac{n\pi}{3}\right) \frac{1}{\left(1 - \left(\frac{n}{3}\right)^2\right)^2} & n \neq 3 \end{cases}$$

* we can use $\sin(A)\sin(B) = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$ *

Computing this directly for the first several values we find

$$E_1 = \frac{\pi^2 \hbar^2}{18 m L^2} \qquad P_1 = \frac{81}{64\pi^2} \approx 0.128$$

$$E_2 = \frac{2\pi^2 \hbar^2}{9 m L^2} \qquad P_2 = \frac{81}{25\pi^2} \approx 0.328$$

$$E_3 = \frac{\pi^2 \hbar^2}{2 m L^2} \qquad P_3 = \frac{1}{3} \approx 0.333$$