Problem Set 5

At this point in the course, we've finished our tour through the foundations of quantum mechanics, and will be moving on to learning techniques to solve quantum mechanical problems that cannot be solved exactly. The goal of this problem set is mainly to set you up for our discussion next week of quantum mechanical perturbation theory, where we will learn to solve problems that are "close" to problems we can solve exactly. You will also review the harmonic oscillator, since many physical systems can be modelled with collections of harmonic oscillators with perturbations. You will then try some simple example problems to give you a basic idea of how perturbation theory works in practice. Finally, there is a question to refresh your memory on working with 1D particle systems.

Problem 0 Read through the "Notes on multipart quantum systems" (posted near the homework on the course webpage). This provides a more formal discussion of how to decribe quantum systems with many parts, such as the collections of qubits used to build a quantum computer.

Problem 1 (Hand in Tuesday)

Provide an approximate solution to the equation

$$x^5 + \frac{x}{100} - 1 = 0 \tag{1}$$

by working through the worksheet handed out in class up to part d. You can think about parts e and f if you like. You don't have to hand in the worksheet, just write out your solution using the method described on the worksheet.

Problem 2 (Hand in Tuesday)

This question is almost exactly the same as the main derivation in Griffiths 6.1, but it's worthwhile working it out on your own. Try it first, then look at Griffiths 6.1 if you get stuck. Suppose we have a Hamiltonian $\hat{H}_0$ with energy eigenstates $|E_0\rangle, |E_1\rangle, \ldots, |E_n\rangle$ such that $\hat{H}_0|E_n\rangle = E_n|E_n\rangle$. If we add a perturbation to this Hamiltonian, so that

$$\hat{H} = \hat{H}_0 + \lambda\hat{\Delta} \tag{2}$$

with λ a parameter, we expect that for small λ, there will be some solution to

$$\hat{H}|\Psi\rangle = E|\Psi\rangle \tag{3}$$

with $|\Psi\rangle$ close to $|E_0\rangle$ and E close to E_0. We can write

$$|\Psi\rangle \quad = \quad |E_0\rangle + \lambda|\delta\Psi\rangle + \ldots$$

$$E = E_0 + \lambda \delta E + \ldots \tag{4}$$

where the dots indicate terms in a Taylor series that are order λ^2 and higher.

a) Plugging (4) and (2) into (3), rewrite the equation as

$$(\qquad) + \lambda(\qquad) + \cdots = 0 \tag{5}$$

Determine the terms in brackets here. These must vanish if the left hand side is zero for all λ.

b) We want to find δE and $|\delta\Psi\rangle$. To do this, it is helpful to write $|\delta\Psi\rangle$ in the $|E_i\rangle$ basis as

$$|\delta\Psi\rangle = c_1|E_1\rangle + \cdots + c_N|E_N\rangle\ . \tag{6}$$

We have not included a term proportional to $|E_0\rangle$ here since the normalization condition $\langle\Psi|\Psi\rangle = 1$ forbids this (adding it would necessarily lengthen the vector). Using the vanishing of the order λ term in (5), find δE and the coefficients c_i. Your answers should be expressed in terms of the operator $\hat{\Delta}$, the states $|E_i\rangle$, and the eigenvalues E_i.

c) Read Griffiths 6.1 if you haven't already.

Problem 3 (Webwork for Thursday; do online - you don't need to hand in a written version)

Review the harmonic oscillator notes from the course website.

a) Using properties of creation and annihilation operators, calculate $\langle 0|aa^\dagger|0\rangle$, $\langle 0|a^2(a^\dagger)^2|0\rangle$ and $\langle 0|a^3(a^\dagger)^3|0\rangle$.

b) Using creation and annihilation operators, calculate $\langle 2|x^2|2\rangle$

Problem 4 (Hand in Thursday) a) For a particle of mass m in an infinite square well located on the interval $[0, L]$, the energy eigenstates are

$$|n\rangle = \int_0^L dx \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)|x\rangle \qquad\qquad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \tag{7}$$

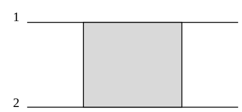
If we are in the ground state and the well suddenly expands to three times its original size, with the right side of the well moving from L to $3L$ (such that the wavefunction doesn't change), and if we immediately measure the energy, what are the probabilities of finding each of the three lowest possible values of energy?

Extra 1

We said that there is a family of physical transformations that can be associated with any Hermitian operator. For the $\hat{X}$ operator in a 1D particle system, let $\hat{T}_X(a)$ be the unitary operators associated with these transformations. If $\psi(x)$ is the wavefunction for a state $|\Psi\rangle$, what is the wavefunction for the state $\hat{T}_X(a)|\Psi\rangle$? How does this state differ physically from the original state?

Extra 2

In a classical computer, we can easily duplicate a bit, i.e. perform an operation using gates so that the value of bit 1 before the operation is stored in both bit 1 and bit 2 after the operation. Show that it is impossible to find a quantum circuit as in the diagram such that if we start with an arbitrary qubit state $|\psi\rangle$ at position 1 and the state $|\downarrow\rangle$ at position 2, the output state will have both qubit 1 and qubit 2 in the state $|\psi\rangle$.



Can you extend your argument to show that with N bits, where the first input bit is $|\psi\rangle$ and the rest are some fixed state of $N-1$ qubits, it's impossible to have a quantum circuit such that both the first and the second output bit will be in state $|\psi\rangle$ for any $|\psi\rangle$?

Extra 3: momentum wavefunctions

The momentum operator has eigenstates $|p\rangle$ which have the property that $\hat{P}|p\rangle = p|p\rangle$. In terms of these, we can express any state as

$$|\Psi\rangle = \int dp \chi(p)|p\rangle \ . \tag{8}$$

where $\chi(p)$ is the *momentum space wavefunction*, equal to

$$\chi(p) = \langle p|\Psi\rangle \tag{9}$$

if we choose the states $|p\rangle$ to be normalized so that $\langle p|q\rangle = \delta(p-q)$. Based on these properties, can you determine the position space wavefunction $\psi_p(x)$ for a momentum eigenstate $|p\rangle$? Given a state with position wavefunction $\psi(x)$, what is the probability that in a measurement of momentum, we will find that the momentum is positive?