PS4 SOLUTIONS: (see worksheet solns. for #1)

② We have: $\hat{H} = -c\,\vec{B}\cdot\vec{S}$

$$= -c\,B_0\,\hat{S}_z$$

So the energy eigenstates will be states with a definite value for S_z. We have:

$$\hat{H}\,|\uparrow\rangle = -cB_0\cdot\frac{\hbar}{2}\,|\uparrow\rangle \quad \text{and} \quad \hat{H}\,|\downarrow\rangle = cB_0\frac{\hbar}{2}\,|\downarrow\rangle \quad (\star)$$

We are given that:

$$|\Psi(t=0)\rangle = |S_x = \tfrac{\hbar}{2}\rangle$$

To find the time evolution, we want to express this in the energy eigenstate basis:

$$|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$$

Now, we can apply the time evolution operator:

$$|\Psi(t)\rangle = e^{-it\hat{H}/\hbar}\,|\Psi(0)\rangle$$

$$= \frac{1}{\sqrt{2}}\left(e^{-it\hat{H}/\hbar}|\uparrow\rangle + e^{-it\hat{H}/\hbar}|\downarrow\rangle\right)$$

using $(\star)$
$$= \frac{1}{\sqrt{2}}\left(e^{\frac{1}{2}icB_0t}\,|\uparrow\rangle + e^{-\frac{1}{2}icB_0t}\,|\downarrow\rangle\right)$$

The eigenstates of $\hat{S}_x$ are $|S_x = \frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ and $|S_x = -\frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$. Thus, if we measure S_x, we could find:

$\frac{\hbar}{2}$ w. prob $|\langle S_x = \frac{\hbar}{2}|\Psi(t)\rangle|^2 = \left|\frac{1}{2}\left(e^{\frac{1}{2}icB_0t} + e^{-\frac{1}{2}icB_0t}\right)\right|^2$

$$= \cos^2\left(\tfrac{1}{2}cB_0t\right)$$

$-\frac{\hbar}{2}$ w. prob $|\langle S_x = -\frac{\hbar}{2}|\Psi(t)\rangle|^2 = \left|\frac{1}{2}\left(e^{\frac{1}{2}icB_0t} - e^{-\frac{1}{2}icB_0t}\right)\right|^2$

$$= |i\sin(\tfrac{1}{2}cB_0t)|^2$$

$$= \sin^2\left(\tfrac{1}{2}cB_0t\right)$$

③

We have an initial state $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$ and Hamiltonian

$$\hat{H} = E_0\begin{pmatrix}9 & 0 & 0\\ 0 & 10 & -4i\\ 0 & 4i & 4\end{pmatrix}$$

To find the time evolution of our state, we'll first determine the eigenstates of $\hat{H}$. We clearly have

$$\hat{H}\begin{pmatrix}1\\0\\0\end{pmatrix} = 9E_0\begin{pmatrix}1\\0\\0\end{pmatrix}$$

The other eigenstates must be orthogonal to this, so they are represented as: $\begin{pmatrix}0\\\alpha\\\beta\end{pmatrix}$ where $\begin{pmatrix}\alpha\\\beta\end{pmatrix}$ must be an eigenvector of the 2×2 matrix $\begin{pmatrix}10 & -4i\\ 4i & 4\end{pmatrix}$

The characteristic polynomial for this is

$$\lambda^2 - 14\lambda + 24 = (\lambda-12)(\lambda-2)$$

So the other eigenvalues of H are 2 and 12.

The eigenvector for eigenvalue λ of a 2×2 matrix $\begin{pmatrix}p & q\\ r & s\end{pmatrix}$ can be taken as: $\begin{pmatrix}q\\ \lambda-p\end{pmatrix}$ as long as this is non-zero. So we have normalized eigenvectors of $\hat{H}$ given by:

$$\vec{V}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 \\ i \\ 2 \end{pmatrix} \qquad \vec{V}_{12} = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 \\ -2i \\ 1 \end{pmatrix}$$

note: these could also be multiplied by a phase

Thus, the eigenvectors for $\hat{H}$ are $\vec{V}_9 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ with $E = 9E_0$, and $\vec{V}_2$ and $\vec{V}_{12}$ with energies $2E_0$ and $12E_0$.

Next we want to express our initial state in terms of the eigenvectors of $\hat{H}$. We have:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = c_9 \vec{V}_9 + c_2 \vec{V}_2 + c_{12} \vec{V}_{12}$$

To find the coefficients, we can use that

$$c_9 = V_9^* \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$c_2 = V_2^* \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \sqrt{\frac{2}{5}}$$

$$c_{12} = V_{12}^* \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{10}}$$

Now we use that energy eigenstates time evolve as $|E_i\rangle \to e^{-iE_i t/\hbar}|E_i\rangle$. So our state after time T is

$$\frac{1}{\sqrt{2}} e^{-9iE_0 t/\hbar} \vec{V}_9 + \sqrt{\frac{2}{5}} e^{-2iE_0 t/\hbar} \vec{V}_2 + \frac{1}{\sqrt{10}} e^{-12iE_0 t/\hbar} \vec{V}_{12}$$

Calling this $|\Psi(T)\rangle$, we now want to calculate

$$P_{S_z=0} = |\langle S_z=0|\Psi(T)\rangle|^2$$

Since we have already expressed everything in the S_z basis from the start, $\langle S_z=0|\Psi(T)\rangle$ is just the second component of the vectors we are using.

This is:

$$\sqrt{\frac{2}{5}}\, e^{-\frac{2iE_0T}{\hbar}} \cdot \frac{i}{\sqrt{5}} + \frac{1}{\sqrt{10}}\, e^{-\frac{12iE_0T}{\hbar}} \cdot \frac{-2i}{\sqrt{5}}$$

$$= \frac{\sqrt{2}}{5} i \left(e^{-\frac{2iE_0T}{\hbar}} - e^{-\frac{12iE_0T}{\hbar}} \right)$$

So: $$P_{S_z=0}(T) = \frac{2}{25} \left| e^{-\frac{2iE_0T}{\hbar}} - e^{-\frac{12iE_0T}{\hbar}} \right|^2$$

$$= \frac{2}{25} \left(e^{-\frac{2iE_0T}{\hbar}} - e^{-\frac{12iE_0T}{\hbar}} \right)\left(e^{\frac{2iE_0T}{\hbar}} - e^{\frac{12iE_0T}{\hbar}} \right)$$

$$= \frac{2}{25} \left(2 - e^{10iE_0T/\hbar} - e^{-10iE_0T/\hbar} \right)$$

$$= \frac{2}{25} \left(2 - 2\cos\left(\frac{10E_0T}{\hbar}\right) \right)$$

$$= \frac{8}{25} \sin^2\left(\frac{5E_0T}{\hbar}\right)$$

Putting in $E_0 = 10^7 s^{-1}\hbar$ and $T = 10^{-8}s$, we get

$$P = \frac{4}{25}(1 - \cos(1)) = 0.0736$$