PROBLEM SET 3 SOLUTIONS

a) We have: $\hat{\theta}\,|\uparrow\rangle = \cos\alpha\,|\uparrow\rangle + i\sin\alpha\,|\downarrow\rangle$

$\hat{\theta}\,|\downarrow\rangle = i\sin\alpha\,|\uparrow\rangle + \cos\alpha\,|\downarrow\rangle$

The matrix representation of this in the $|\uparrow\rangle, |\downarrow\rangle$ basis is

$$\theta = \begin{pmatrix} \cos\alpha & i\sin\alpha \\ i\sin\alpha & \cos\alpha \end{pmatrix}$$

We have:

$$\theta^{+}\theta = \begin{pmatrix} \cos\alpha & -i\sin\alpha \\ -i\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\alpha & i\sin\alpha \\ i\sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\alpha + \sin^2\alpha & 0 \\ 0 & \cos^2\alpha + \sin^2\alpha \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so the operator is unitary.

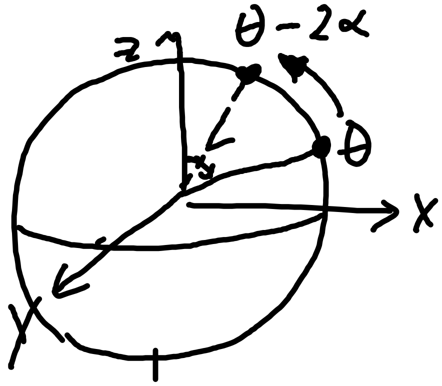
b) Acting on $\cos\frac{\theta}{2}\,|\uparrow\rangle + \sin\frac{\theta}{2}\,|\downarrow\rangle$, we get:

$$\cos\tfrac{\theta}{2}\left(\cos\alpha\,|\uparrow\rangle - \sin\alpha\,|\downarrow\rangle\right) + \sin\tfrac{\theta}{2}\left(\sin\alpha\,|\uparrow\rangle + \cos\alpha\,|\downarrow\rangle\right)$$

$$= \left(\cos\tfrac{\theta}{2}\cos\alpha + \sin\tfrac{\theta}{2}\sin\alpha\right)|\uparrow\rangle + \left(\sin\tfrac{\theta}{2}\cos\alpha - \cos\tfrac{\theta}{2}\sin\alpha\right)|\downarrow\rangle$$

$$= \cos\left(\tfrac{\theta}{2} - \alpha\right)|\uparrow\rangle + \sin\left(\tfrac{\theta}{2} - \alpha\right)|\downarrow\rangle$$

So: $\theta' = \theta - 2\alpha$.



The transformation corresponds to a 2θ rotation about the y-axis.

For small α, we can expand the matrix for O as:

$$\begin{pmatrix} 1-\frac{\alpha^2}{2}+\dots & \alpha+\dots \\ -\alpha+\dots & 1-\frac{\alpha^2}{2}+\dots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\alpha \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \dots$$

order α^2 & higher.

So if $\hat{O} = \mathbb{1} + i\alpha\hat{a} + \dots$, we have that the matrix representing a is $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. This is Hermitian and is proportional to the spin y operator.

② We have:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

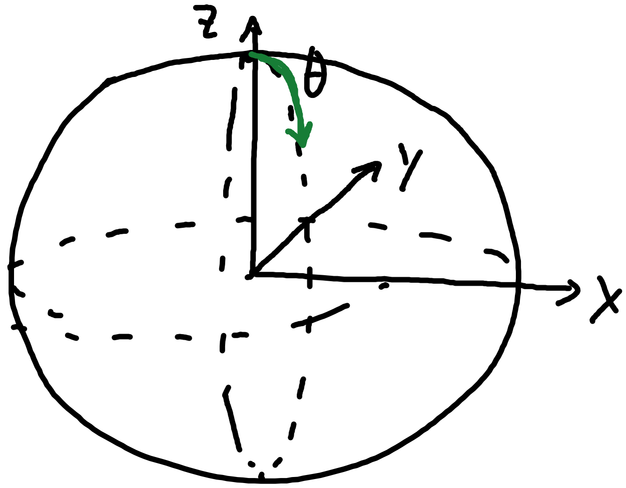
To find e^{iaX}, we first use the power series method. Since $X^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, all even powers of X are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and all odd powers are $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$\begin{aligned} e^{iaX} &= \mathbb{1} - \frac{1}{2} a^2 X^2 + \frac{1}{4!} a^4 X^4 + \cdots \\ &\quad + iaX + \frac{1}{3!} (iaX)^3 + \frac{1}{5!} (iaX)^5 + \cdots \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left(1 - \frac{1}{2} a^2 + \frac{1}{4!} a^4 - \cdots \right) \\ &\quad + i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(a - \frac{1}{3!} a^3 + \frac{1}{5!} a^5 - \cdots \right) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos a + i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin a \\ &= \begin{pmatrix} \cos a & i \sin a \\ i \sin a & \cos a \end{pmatrix} \end{aligned}$$

Acting on $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, this gives $\begin{pmatrix} \cos a \\ i \sin a \end{pmatrix}$ so we have

$$e^{iaX} |\uparrow\rangle = \cos\left(\frac{\theta}{2}\right) |\uparrow\rangle + e^{i\varphi} \sin\left(\frac{\theta}{2}\right) |\downarrow\rangle$$

with $\theta = 2a$, $\varphi = \frac{\pi}{2}$.



We see that this is a rotation about the y-axis.

Extra problem 1

solution: First, suppose that both operators are represented as diagonal matrices in some basis $|n\rangle$.

Then $\hat{a}|n\rangle = a_n|n\rangle$ and $\hat{B}|n\rangle = b_n|n\rangle$, so:

$$(\hat{a}\hat{B}-\hat{B}\hat{a})|n\rangle = \hat{a}\hat{B}|n\rangle - \hat{B}\hat{a}|n\rangle$$
$$= a_n b_n|n\rangle - b_n a_n|n\rangle = 0$$

For any other state $|\psi\rangle = \sum c_n|n\rangle$, we have

$$(\hat{a}\hat{B}-\hat{B}\hat{a})|\psi\rangle = \sum c_n(\hat{a}\hat{B}-\hat{B}\hat{a})|n\rangle = 0$$ so $\hat{a}$ and $\hat{B}$ commute.

Conversely, suppose that $\hat{a}$ and $\hat{B}$ commute. In a basis of eigenstates for the operator $\hat{a}$, we have $\hat{a}|n\rangle = a_n|n\rangle$. Generally, we must have $\hat{B}|n\rangle = \sum_m b_{mn}|m\rangle$. Now since $\hat{a}$ and $\hat{B}$ commute, we get

$$(\hat{a}\hat{B}-\hat{B}\hat{a})|n\rangle = 0$$
$$\Rightarrow \hat{a}\sum_m b_{mn}|m\rangle - \hat{B}a_n|n\rangle = 0$$
$$\Rightarrow \sum_m b_{mn}(a_m - a_n)|m\rangle = 0$$
$$\Rightarrow b_{mn}(a_m - a_n) = 0 \text{ for every } m$$

if a vector is zero, the coefficients for the vector in any basis are zero.

Assuming a_m are all distinct, this implies $b_{mn}=0$ for $m \neq n$ so $\hat{B}$ is also diagonal in this basis.

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If the a_m are not all distinct, then we can only that A and B take the form

$$A = \begin{pmatrix} \underbrace{a_1 \cdots a_1}_{n_1 \text{ equal}} & & & \\ & \underbrace{a_2 \cdots a_2}_{n_2 \text{ equal}} & & \\ & & \ddots & \\ & & & a_n \cdots a_n \end{pmatrix} \qquad B = \begin{pmatrix} \underbrace{B_1}_{n_1 \times n_1} & & & \\ & \underbrace{B_2}_{n_2 \times n_2} & & \\ & & \ddots & \\ & & & B_n \end{pmatrix}$$

Here, we have assumed that the basis is ordered so that eigenvectors with the same eigenvalue appear together. Now, consider the subspace of vectors that have eigenvalue a_i. Acting on this subspace, A acts as a_i times the identity matrix, while B acts as the $n_i \times n_i$ Hermitian matrix B_i. Any vector in this subspace is an A eigenvector with eigenvalue a_i, so we can replace our original eigenvectors in this subspace with the eigenvectors of B_i. If we similarly do this for all the subspaces, the new set of eigenvectors will all be eigenvectors of both A and B, so A an B are both diagonal in this basis.

EXTRA #2:

Since $[\hat{O}, \hat{H}] = 0$:

we can find a basis $|\lambda_n\rangle$ where $\hat{H}$ and $\hat{O}$ are both diagonal, with $\hat{O}|\lambda_n\rangle = \lambda_n|\lambda_n\rangle$ and $\hat{H}|\lambda_n\rangle = E_n|\lambda_n\rangle$. Then:

$$\frac{d}{dt} P_n = \frac{d}{dt} |\langle\lambda_n|\psi\rangle|^2$$

$$= \frac{d}{dt} \langle\psi|\lambda_n\rangle\langle\lambda_n|\psi\rangle$$

$$= \frac{i}{\hbar} \langle\psi|\hat{H}|\lambda_n\rangle\langle\lambda_n|\psi\rangle - \frac{i}{\hbar} \langle\psi|\lambda_n\rangle\langle\lambda_n|\hat{H}|\psi\rangle$$

$$= \frac{i}{\hbar} \langle\psi|\lambda_n|\lambda_n\rangle\langle\lambda_n|\psi\rangle - \frac{i}{\hbar} \langle\psi|\lambda_n\rangle\langle\lambda_n|\lambda_n|\psi\rangle$$

$$= 0$$