Problem Set 3

In this problem set, you will get some hands on experience seeing that a Hermitian operator can be associated with any infinitesimal unitary transformation. Going the other way, you will see that by starting from an infinitesimal transformation associated with a Hermitian operator, we can build up a family of unitary transformations by acting with this many times. The first question is a bit of prep for next class.

Problem 0 (hand in Tuesday, participation credit)

Next class, we're going talk more about symmetries and conservation laws in quantum mechanics. Take a few minutes to think about and jot down answers to the following questions:

What is meant by a symmetry in classical physics? Can you think of a way to define what would be meant by a symmetry in quantum mechanics?

Problem 1 (Webwork for Thursday)

a) To make sure you understand the definition of an operator being unitary, decide whether the operator represented by the following matrix is unitary or not:

$$\begin{aligned} \hat{\mathcal{O}}|\uparrow\rangle &= \cos(\alpha)|\uparrow\rangle + i\sin(\alpha)|\downarrow\rangle \\ \hat{\mathcal{O}}|\downarrow\rangle &= i\sin(\alpha)|\uparrow\rangle + \cos(\alpha)|\downarrow\rangle \end{aligned}$$

b) In class, we mentioned the family of unitary operators represented by matrices

$$\begin{aligned} \hat{\mathcal{O}}|\uparrow\rangle &= \cos(\alpha)|\uparrow\rangle - \sin(\alpha)|\downarrow\rangle \\ \hat{\mathcal{O}}|\downarrow\rangle &= \sin(\alpha)|\uparrow\rangle + \cos(\alpha)|\downarrow\rangle \end{aligned}$$

In order to get an idea of what physical transformation this corresponds to, write the result of applying this transformation to the state

$$\cos(\theta/2)|\uparrow\rangle + \sin(\theta/2)|\downarrow\rangle \ . \tag{1}$$

The result can be expressed as

$$\cos(\theta'/2)|\uparrow\rangle + \sin(\theta'/2)|\downarrow\rangle \ . \tag{2}$$

What is θ' in terms of α and θ? What physical transformation does this correspond to?

c) If we call this operator $\hat{T}(\alpha)$, and consider small α, we can write

$$\hat{T}(\theta) = \mathbf{1} + i\alpha\hat{\mathcal{A}} + \dots \ . \tag{3}$$

Find the matrix representing the operator $\hat{\mathcal{A}}$ in the $|\uparrow\rangle$, $|\downarrow\rangle$ basis. We already did this in class; this is just a review. Is it Hermitian? Do you recognize the operator $\hat{\mathcal{A}}$?

Problem 2 (Webwork for Thursday)

In class, I mentioned that we can get a "big" physical transformation by acting with a small one many times. Let's see how that works. Suppose we have

$$\hat{T}(\epsilon) = \mathbf{1} + i\epsilon\hat{\mathcal{A}} + \ldots \tag{4}$$

We want to consider the limit $\epsilon \to 0$ to define an infinitesimal transformation, but act with this a very large number of times in order to get something finite. A way to do this is to let $\epsilon = a/N$ and act N times, then take the limit $N \to \infty$. This gives:

$$\lim_{N\to\infty} (\mathbf{1} + i\frac{a}{N}\hat{\mathcal{A}} + \ldots)^N = \lim_{N\to\infty} \left(\mathbf{1} + \frac{ia\hat{\mathcal{A}}}{N}\right)^N = e^{ia\hat{\mathcal{A}}} \tag{5}$$

Here, in the first step, we have discarded the terms at higher orders in $\epsilon = x/N$. We can check that if we kept them, the extra terms we would get vanish in the limit $N \to \infty$. In the second step, we have used the famous limit formula for the exponential. But what does the exponential of an operator mean? One way to define it is by a power series:

$$e^{\hat{\mathcal{O}}} = 1 + \hat{\mathcal{O}} + \frac{1}{2}\hat{\mathcal{O}}^2 + \ldots \tag{6}$$

Another way is to work in a basis where $\hat{\mathcal{O}}$ is diagonal. In this case, if

$$\hat{\mathcal{O}} = \sum_n \lambda_n |\lambda_n\rangle\langle\lambda_n| \tag{7}$$

Then

$$e^{\hat{\mathcal{O}}} = \sum_n e^{\lambda_n} |\lambda_n\rangle\langle\lambda_n| \tag{8}$$

Using one of these methods, calculate the matrix representation of

$$e^{iaX} \tag{9}$$

in the $|\uparrow\rangle$, $|\downarrow\rangle$ basis, where X is an operator represented by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{10}$$

in this basis. If we write $e^{iaX}|\uparrow\rangle$ in the form

$$\cos(\theta/2)|\uparrow\rangle + e^{i\phi}\sin(\theta/2)|\downarrow\rangle\ , \tag{11}$$

what are θ and ϕ in terms of a? Based on this, can you say what this transformation is using the sphere picture?

EXTRA 1 (not to be handed in)

If $\hat{A}$ and $\hat{B}$ are Hermitian operators, show that they commute if and only if there is a basis where they are both represented by a diagonal matrix. (Hint: you may want to start with the case where the eigenvalues of $\hat{A}$ are distinct, and also work in a basis where the matrix for A is diagonal.

EXTRA 2 (not to be handed in)

Show that if $[\hat{O}, \hat{H}] = 0$ in a system with a time-independent Hamiltonian $\hat{H}$, then for any state $|\Psi(t)\rangle$ satisfying the Schrödinger equation, all probabilities p_n for measuring the different eigenvalues of $\mathcal{O}$ are unchanging in time. Since we already showed that $\mathcal{O}$ having constant expectation values for all states implies $[\hat{O}, \hat{H}] = 0$, this shows that $\mathcal{O}$ having constant expectation values for all states implies that any state will also have constant probabilities for $\mathcal{O}$.