PROBLEM SET 2 SOLUTIONS

WEBWORK

1 i) We have for the state $|\Psi\rangle$ that the probabilities for the possible values of $\mathcal{O}$ are:

$$P_{-1} = \frac{1}{4}, \quad P_0 = \frac{1}{4}, \quad P_1 = \frac{1}{2}$$

Then the expectation value of $\mathcal{O}$ is:

$$\begin{aligned}
\langle \mathcal{O} \rangle &= \sum_n \lambda_n P_n \\
&= 1 \cdot P_1 + 0 \cdot P_0 + (-1) \cdot P_{-1} \\
&= \frac{1}{2} - \frac{1}{4} \\
&= \frac{1}{4}
\end{aligned}$$

The uncertainty is:

$$\begin{aligned}
\Delta \mathcal{O} = \left(\sum_n (\lambda_n - \langle \mathcal{O} \rangle)^2 P_n \right)^{\frac{1}{2}} &= \left(\left(1 - \frac{1}{4}\right)^2 \cdot \frac{1}{2} + \left(0 - \frac{1}{4}\right)^2 \cdot \frac{1}{4} + \left(-1 - \frac{1}{4}\right)^2 \cdot \frac{1}{4} \right)^{\frac{1}{2}} \\
&= \frac{\sqrt{11}}{4}
\end{aligned}$$

ii)

$$\begin{aligned}
[Z, X] &= ZX - XZ \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}
\end{aligned}$$

#2 We are given that:

$$|\uparrow\rangle \rightarrow \cos\theta|\uparrow\rangle - \sin\theta|\downarrow\rangle \qquad |\downarrow\rangle \rightarrow +\sin\theta|\uparrow\rangle + \cos\theta|\downarrow\rangle$$

For a state $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$, we have

$$\begin{aligned}\hat{O}|\psi\rangle &= a\hat{O}|\uparrow\rangle + b\hat{O}|\downarrow\rangle \\ &= a(\cos\theta|\uparrow\rangle - \sin\theta|\downarrow\rangle) \\ &\quad + b(\sin\theta|\uparrow\rangle + \cos\theta|\downarrow\rangle) \\ &= (a\cos\theta + b\sin\theta)|\uparrow\rangle + (-a\sin\theta + b\cos\theta)|\downarrow\rangle\end{aligned}$$

So we can represent the action of the operator in the $|\uparrow\rangle, |\downarrow\rangle$ basis as

$$\begin{pmatrix} a \\ b \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

If a state represented by $\begin{pmatrix} a \\ b \end{pmatrix}$ is an eigenvector with eigenvalue λ, we have:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}$$

or

$$\begin{pmatrix} \cos\theta - \lambda & \sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

If this is true for nonzero a, b the matrix must have determinant zero, so:

$$\lambda^2 - 2\lambda\cos\theta + 1 = 0$$

Same as just writing Characteristic polynomial

Solving, we have: $\lambda = \cos\theta \pm \sqrt{\cos^2\theta - 1}$

$$= \cos\theta \pm i\sin\theta = e^{i\theta}, e^{-i\theta}$$

We can now find the eigenvector for each eigenvalue, starting from *. For $\lambda = e^{i\theta} = \cos + i\sin\theta$, we get

$$\begin{pmatrix} -i\sin\theta & \sin\theta \\ -\sin\theta & -i\sin\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

So: $\begin{pmatrix} a \\ b \end{pmatrix} = c \cdot \begin{pmatrix} 1 \\ +i \end{pmatrix}$

This corresponds to a normalized state

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle + i|\downarrow\rangle \right)$$

Similarly, for $\lambda = e^{-i\theta} = \cos\theta - i\sin\theta$, we find a normalized eigenstate.

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle - i|\downarrow\rangle \right)$$

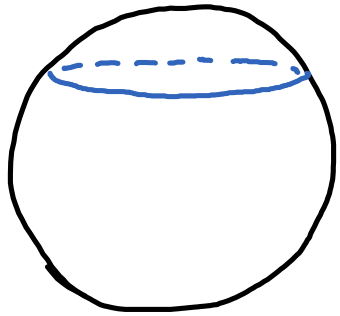
Worksheet solutions

① Let's call the eigenstates in the Z basis $|-1\rangle$ and $|1\rangle$. Then we can write the most general state as

$$|\Psi\rangle = z_1 |1\rangle + z_{-1} |-1\rangle$$

We can assume $|z_1|^2 + |z_{-1}|^2 = 1$ and that z_1 is real and positive. From the experiment, about $3/4$ of the Z measurements are $+1$, so $P_1 \approx 3/4$ and $P_{-1} = \frac{1}{4}$. This tells us $|z_1|^2 = 3/4$, $|z_2|^2 = \frac{1}{4}$, so $z_1 = \frac{\sqrt{3}}{2}$ (we assumed it's real & positive) and $z_2 = \frac{1}{2} e^{i\varphi}$, but we don't know what φ is yet.

Graphically, we have narrowed down the possible quantum state to a particular circle on the sphere that describes our most general state.



Next, we want to use the X and Y measurements to constrain the state further. From our state $|\psi\rangle = \frac{\sqrt{3}}{2}|1\rangle + \frac{1}{2} e^{i\varphi}|-1\rangle$, we can now compute either $\langle X \rangle$ or $P_{X=1}$ and $P_{X=-1}$ and compare with our experimental results.

One way forward is to find the eigenstates for X in order to calculate the probabilities for the different possible X measurements. The matrix representation for X in the Z basis is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ so we find eigenvalues ± 1 and normalized eigenvectors $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $X = +1$ and

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for $X = -1$. For our state $|\Psi\rangle$, the probability of measuring $X = 1$ is

$$P_{X=1} = |\langle X=1|\Psi\rangle|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle 1| + \tfrac{1}{\sqrt{2}}\langle -1|\right)|\Psi\rangle\right|^2$$

$$= \left|\frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} e^{i\varphi}\right|^2$$

or:

$$\left|\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right)^* \cdot \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} e^{i\varphi} \end{pmatrix}\right|^2$$

$$= \left(\frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\cos\varphi\right)^2 + \left(\frac{1}{2\sqrt{2}}\sin\varphi\right)^2$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{4}\cos\varphi$$

Since 87.5% measured as $X = 1$, we have

$$\frac{7}{8} = \frac{1}{2} + \frac{\sqrt{3}}{4}\cos\varphi$$

$$\Rightarrow \cos\varphi = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \varphi = \frac{\pi}{6} \text{ or } \frac{11\pi}{6} \left(\equiv -\frac{\pi}{6}\right)$$

Thus, we have 2 possible states:

$$|\Psi_\pm\rangle = \frac{\sqrt{3}}{2}|1\rangle + e^{\pm i\frac{\pi}{6}}\frac{1}{2}|-1\rangle$$

Another way to do this part would have been to say that the expectation value of X is

$$\bar{X} = \frac{7}{8}\cdot 1 + \frac{1}{8}\cdot(-1) = \frac{3}{4}$$ ←from the experiment

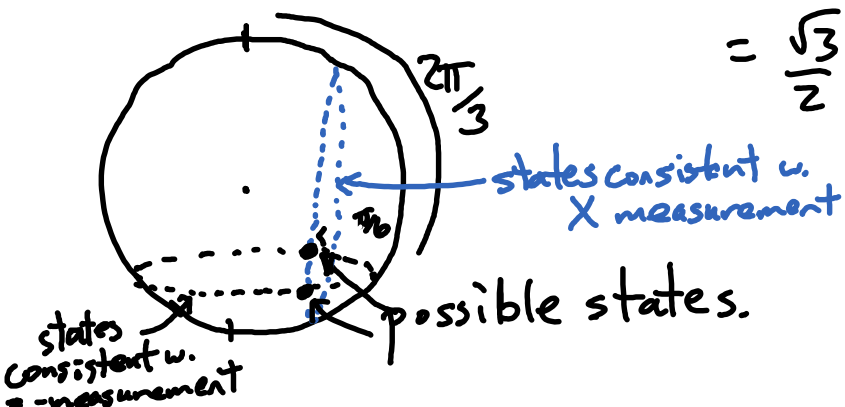
and then use

$$\bar{X} = \langle\Psi|\cdot\hat{X}|\Psi\rangle = \left(\frac{\sqrt3}{2}, \frac12 e^{i\varphi}\right)^{*}\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\frac{\sqrt3}{2}\\ \frac12 e^{i\varphi}\end{pmatrix}$$

$$= \left(\frac{\sqrt3}{2}, \frac12 e^{-i\varphi}\right)\begin{pmatrix}\frac12 e^{i\varphi}\\ \frac{\sqrt3}{2}\end{pmatrix}$$

$$= \frac{\sqrt3}{4}\left(e^{i\varphi}+e^{-i\varphi}\right)$$

$$= \frac{\sqrt3}{2}\cos(\varphi) \quad \text{so:} \quad \frac{\sqrt3}{2}\cos\varphi = \frac34$$



We can distinguish thes by using a Y detector: we have:

$$\langle\Psi_\pm|\hat{Y}|\Psi_\pm\rangle = \left(\frac{\sqrt3}{2}, \frac12 e^{\mp i\frac{\pi}{6}}\right)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\frac{\sqrt3}{2}\\ \frac12 e^{\pm i\frac{\pi}{6}}\end{pmatrix}$$

$$= \frac{\sqrt3}{2}\,\frac{e^{\pm i\frac{\pi}{6}} - e^{\mp i\frac{\pi}{6}}}{2i}$$

$$= \pm\frac{\sqrt3}{2}\sin\left(\frac{\pi}{6}\right) = \pm\frac{\sqrt3}{4}$$

we find more $Y=+1$s, so we have $\varphi = +\frac{\pi}{6}$

★$\langle Y\rangle$ is different for the two possible states★

$$\boxed{\text{Conclusion: } |\psi\rangle \approx \frac{\sqrt3}{2}|\uparrow\rangle + \frac12\left(\frac{\sqrt3}{2}+\frac{i}{2}\right)|\downarrow\rangle}$$

Jan 17 (2) In the Z_3 basis, the observable X_3 is represented by:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

We need to determine the form of the X_3 eigenstates. We are given that the eigenvalues are $-1, 0, 1$. For $\lambda_x = 0$, the state must satisfy

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} b \\ a+c \\ b \end{pmatrix} = 0$$

So we can take the normalized eigenvector as $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

Particles that measure as $X=0$ will have this state after they pass through the detector. This is

$$\frac{1}{\sqrt{2}}\,|Z=1\rangle - \frac{1}{\sqrt{2}}\,|Z=-1\rangle$$

so we'll have 50% $Z=1$ and 50% $Z=-1$ in the second detector.

For $\lambda_x = 1$, the state must satisfy

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} b \\ a+c \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

So we find a normalized eigenvector $\frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$ and the Z measurements will give 25% $Z=1$, 25% $Z=-1$, and 50% $Z=0$.

For $\lambda_x = 1$, the state must satisfy

$$\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = -\begin{pmatrix}a\\b\\c\end{pmatrix} \Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}b\\a+c\\b\end{pmatrix} = -\begin{pmatrix}a\\b\\c\end{pmatrix}$$

so we find a normalized eigenvector $\frac{1}{2}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}$ and the Z measurements will give 25% $Z=1$, 25% $Z=-1$ and 50% $Z=0$. Thus, the table is:

X measurement	% $Z=-1$	% $Z=0$	% $Z=1$
-1	25	50	25
0	50	0	50
1	25	50	25.

③ Let $|\Psi\rangle = \sum c_n |\lambda_n\rangle$ where $|\lambda_n\rangle$ are the eigenvectors for $\hat{O}$. Then:

$$\hat{O}|\Psi\rangle = \sum_n c_n \hat{O}|\lambda_n\rangle$$
$$= \sum_n c_n \lambda_n |\lambda_n\rangle$$

$$\hat{O}^2|\Psi\rangle = \sum_n c_n \lambda_n \hat{O}|\lambda_n\rangle$$
$$= \sum_n c_n \lambda_n^2 |\lambda_n$$

$$\langle\Psi|\hat{O}|\Psi\rangle = \left(\sum_m c_m^* \langle\lambda_m|\right)\left(\sum_n c_n \lambda_n |\lambda_n\rangle\right)$$
$$= \sum_{m,n} c_m^* c_n \lambda_n \langle\lambda_m|\lambda_n\rangle$$
$$= \sum_n c_n^* c_n \lambda_n$$

since $\langle\lambda_m|\lambda_n\rangle = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$

by orthonormality.

Thus: $$\langle\Psi|\hat{O}|\Psi\rangle = \sum_n |c_n|^2 \lambda_n$$
$$= \sum_n p_n \lambda_n$$
$$= \langle O\rangle$$

Similarly:

$$\langle \Psi | \hat{O}^2 | \Psi \rangle = \left(\sum_m c_m^* \langle \lambda_m |\right)\left(\sum_n c_n \lambda_n^2 | \lambda_n \rangle\right)$$

$$= \sum c_n^* c_n \lambda_n^2$$

$$= \sum_n p_n \lambda_n^2$$

Finally: $\langle \Psi | \hat{O}^2 | \Psi \rangle - \langle \Psi | \hat{O} | \Psi \rangle^2$

$$= \sum_n p_n \lambda_n^2 - \left(\sum_n p_n \lambda_n\right)^2 \quad (\star)$$

The uncertainty ΔO satisfies

$$(\Delta O)^2 = \sum_n p_n \left(\lambda_n - \langle O \rangle\right)^2$$

$\sum_n p_n = 1$

$$= \sum_n p_n \lambda_n^2 - \sum_n 2 p_n \lambda_n \langle O \rangle + \sum_n p_n \langle O \rangle^2$$

$$= \sum_n p_n \lambda_n^2 - 2 \langle O \rangle \langle O \rangle + \langle O \rangle^2$$

$$= \sum_n p_n \lambda_n^2 - \left(\sum_n p_n \lambda_n\right)^2$$ which matches with $(\star)$