PROBLEM SET 1 SOLUTIONS

① We are given that

$$|A\rangle = \frac{1}{2}|\uparrow\rangle - i\frac{\sqrt{3}}{2}|\downarrow\rangle$$

$$|B\rangle = \frac{i}{2}|\uparrow\rangle + \frac{\sqrt{3}}{2}|\downarrow\rangle$$

We recall that to compute $\langle\psi_1|\psi_2\rangle$, we need to complex conjugate the coefficients in $|\psi_1\rangle$. We can write this as:

$$\langle A|A\rangle = \left(\frac{1}{2}\langle\uparrow| + i\frac{\sqrt{3}}{2}\langle\downarrow|\right)\left(\frac{1}{2}|\uparrow\rangle - i\frac{\sqrt{3}}{2}|\downarrow\rangle\right)$$

$$= \frac{1}{4}\langle\uparrow|\uparrow\rangle - i\frac{\sqrt{3}}{4}\langle\downarrow|\uparrow\rangle + \frac{i\sqrt{3}}{4}\langle\uparrow|\downarrow\rangle + \frac{3}{4}\langle\downarrow|\downarrow\rangle$$

$$= 1$$

using the orthonormality $\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1$

$\langle\uparrow|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 0$

Similarly:

$$\langle B|B\rangle = \left(-\frac{i}{2}\langle\uparrow| + \frac{\sqrt{3}}{2}\langle\downarrow|\right)\left(\frac{i}{2}|\uparrow\rangle + \frac{\sqrt{3}}{2}|\downarrow\rangle\right)$$

$$= \frac{1}{4}\langle\uparrow|\uparrow\rangle + \frac{3}{4}\langle\downarrow|\downarrow\rangle = 1$$

$$\langle A|B\rangle = \left(\frac{1}{2}\langle\uparrow| + i\frac{\sqrt{3}}{2}\langle\downarrow|\right)\left(\frac{i}{2}|\uparrow\rangle + \frac{\sqrt{3}}{2}|\downarrow\rangle\right)$$

$$= \frac{i}{4}\langle\uparrow|\uparrow\rangle + i\frac{3}{4}\langle\downarrow|\downarrow\rangle = i$$

Finally, $\langle B|A\rangle = \langle A|B\rangle^* = -i$

② Given $|\psi\rangle = i|\uparrow\rangle - \sqrt{3}|\downarrow\rangle$ we first note that $\langle\psi|\psi\rangle = |i|^2 + |-\sqrt{3}|^2 = 4$ so taking $|\psi\rangle \to \frac{1}{2}|\psi\rangle$ ensures that our state will be normalized properly. This gives

$$\frac{i}{2}|\uparrow\rangle - \frac{\sqrt{3}}{2}|\downarrow\rangle$$

Next, we multiply by a "phase" (i.e. a cplx number with norm 1) to make the first coefficient real and positive. We take

$$|\psi\rangle \to -i|\psi\rangle = \frac{1}{2}|\uparrow\rangle + \frac{\sqrt{3}i}{2}|\downarrow\rangle$$

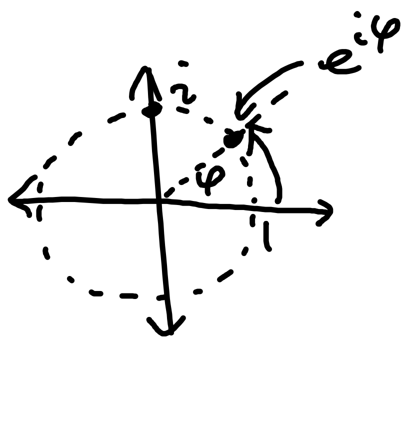
③ The new state should now be in the form

$$\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle$$

for some $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$.

We need $\cos\frac{\theta}{2} = \frac{1}{2}$ so $\theta = \frac{2\pi}{3}$.

Also: $e^{i\varphi} = i$ so $\varphi = \frac{\pi}{2}$



(4) We have $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle$

If we measure S_z, we have probability $\left|\cos\frac{\theta}{2}\right|^2 = \cos^2\frac{\theta}{2}$ to find $\frac{\hbar}{2}$ and probability $\left|e^{i\varphi}\sin\left(\frac{\theta}{2}\right)\right|^2 = \sin^2\frac{\theta}{2}$ to find $-\frac{\hbar}{2}$, so the expectation value of S_z in our state is

$$\langle S_z\rangle = \cos^2\frac{\theta}{2}\cdot\frac{\hbar}{2} + \sin^2\frac{\theta}{2}\left(-\frac{\hbar}{2}\right)$$
$$= \frac{\hbar}{2}\cos\theta$$

To find $\langle S_x\rangle$, we should write

$$|\psi\rangle = a\left|S_x = \frac{\hbar}{2}\right\rangle + b\left|S_x = -\frac{\hbar}{2}\right\rangle \qquad (+)$$

To find a and b, we can use that

$$\left|S_x = \frac{\hbar}{2}\right\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$$
$$\text{and } \left|S_x = -\frac{\hbar}{2}\right\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right) \qquad (*)$$

Since these are orthonormal, we have that

$$a = \left\langle S_x = \frac{\hbar}{2}\middle|\psi\right\rangle = \frac{1}{\sqrt{2}}\left(\langle\uparrow| + \langle\downarrow|\right)\left(\cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}e^{i\varphi}|\downarrow\rangle\right)$$
$$= \frac{1}{\sqrt{2}}\cos\frac{\theta}{2} + \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}e^{i\varphi}$$

$$b = \left\langle S_x = -\frac{\hbar}{2}\middle|\psi\right\rangle = \frac{1}{\sqrt{2}}\cos\frac{\theta}{2} - \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}e^{i\varphi}$$

(or we can use to find $|\uparrow\rangle = \frac{1}{\sqrt{2}}\left(\left|S_x = \frac{\hbar}{2}\right\rangle + \left|S_x = -\frac{\hbar}{2}\right\rangle\right)$
$|\downarrow\rangle = \frac{1}{\sqrt{2}}\left(\left|S_x = \frac{\hbar}{2}\right\rangle - \left|S_x = -\frac{\hbar}{2}\right\rangle\right)$
and plug these in to our state to rewrite it in the form (+))

Either way, we have:

$$P_{S_x=\frac{\hbar}{2}} = |a|^2 = a\cdot a^* = \frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2} + e^{i\varphi}\sin\frac{\theta}{2}\right) \cdot \frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2} + e^{-i\varphi}\sin\frac{\theta}{2}\right)$$

$$= \frac{1}{2}\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + \cos\frac{\theta}{2}\sin\frac{\theta}{2}\left(e^{i\varphi}+e^{-i\varphi}\right)\right)$$

$$= \frac{1}{2}\left(1 + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\varphi\right)$$

$$= \frac{1}{2}\left(1 + \sin\theta\cos\varphi\right)$$

* recall

$$\frac{e^{i\varphi}+e^{-i\varphi}}{2} = \cos\theta$$

$$\frac{e^{i\varphi}-e^{-i\varphi}}{2i} = \sin\theta$$

$$P_{S_x=-\frac{\hbar}{2}} = |b|^2 = b\cdot b^* = \frac{1}{2}\left(1-\sin\theta\cos\varphi\right)$$

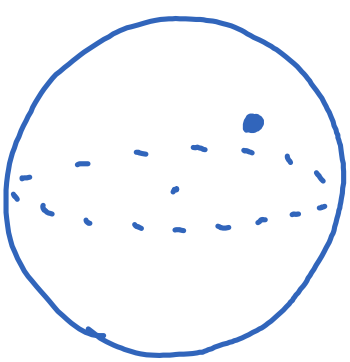
$$\langle S_x \rangle = \frac{\hbar}{2}P_{S_x=\frac{\hbar}{2}} - \frac{\hbar}{2}P_{S_x=-\frac{\hbar}{2}} = \frac{\hbar}{2}\cdot\sin\theta\cos\varphi$$

Similarly, we find

$$|\psi\rangle = c\left|S_y=\frac{\hbar}{2}\right\rangle + d\left|S_y=-\frac{\hbar}{2}\right\rangle$$

with $c = \langle S_y=\frac{\hbar}{2}|\psi\rangle$ $d = \langle S_y=-\frac{\hbar}{2}|\psi\rangle$. Following the same steps as for S_x, we get:

$$\langle S_y \rangle = \frac{\hbar}{2}\sin\theta\sin\varphi$$



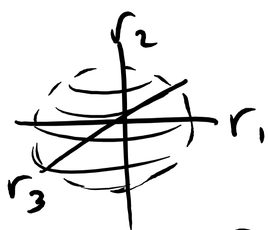
* We have seen that $(\langle S_x\rangle, \langle S_y\rangle, \langle S_z\rangle)$ are exactly the cartesian coordinates of the point on the sphere associated w. $|\psi\rangle$, where we take the radius to be $\hbar/2$ *

(X) For a 3D Hilbert space, we can write any state as:

$$z_1|1\rangle + z_2|2\rangle + z_3|3\rangle$$

It is useful to write $z_1 = r_1 e^{i\varphi_1}$, $z_2 = r_2 e^{i\varphi_2}$, $z_3 = r_3 e^{i\varphi_3}$. Then, by multiplication with an overall real number, we can always normalize the state. This gives an equivalent state with

$$r_1^2 + r_2^2 + r_3^2 = 1$$



These lengths are the the points on a unit sphere in $\mathbb{R}^3$, so we can represent them using spherical coords:

$$r_1 = \cos\theta, \; r_2 = \sin\theta\cos\phi, \; r_3 = \sin\theta\sin\phi$$

with $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \phi \leq \pi/2$ so that all r's are positive. We can also multiply by a phase to set φ_1 to 0. This gives

$$|\psi\rangle = \cos\theta|1\rangle + \sin\theta\cos\phi\, e^{i\varphi_1}|2\rangle + \sin\theta\sin\phi\, e^{i\varphi_2}|3\rangle$$

$0 \leq \varphi_1 \leq 2\pi$

$0 \leq \varphi_2 \leq 2\pi$

* see next page *

Since the 2D case gave an ordinary sphere, you may be wondering if this is some kind of higher dimensional sphere we are describing. Actually, the space of inequivalent quantum states for an N-dimensional Hilbert space is a $2N-2$ dimensional space called $\mathbb{CP}^{N-1}$. This is the same as the 2D sphere S^2 for $N=2$, but is not a sphere for $N>2$.

Mathematically, $\mathbb{CP}^{N-1}$ is defined as a space of "equivalence classes" of N-tuples

$$(z_1, \ldots, z_N)$$

of complex numbers where we require that at least one of the z_i s is nonzero and we declare $(z_1, \ldots, z_N)$ and $(\hat{z}_1, \ldots, \hat{z}_N)$ to be equivalent if

$$(\hat{z}_1, \ldots, \hat{z}_N) = (\lambda z_1, \ldots, \lambda z_N)$$

for some nonzero complex number λ.

EXERCISE: show that each equivalence class with nonzero z_1 has a unique representative with $|z_1|^2 + \ldots + |z_N|^2 = 1$ and z_1 real.