Problem set 11 solutions

① We have $\vec{E}=0$ and $\vec{B}=(0,0,B)$. To write the Hamiltonian for a charged particle in this field, we first need to find a potential description. We can take $\phi=0$ and $\vec{A}$ time-independent such that

$$\vec{\nabla}\times\vec{A}=\vec{B}$$

$$\Rightarrow \partial_x A_y - \partial_y A_x = B$$

$$\partial_x A_z - \partial_z A_x = 0$$

$$\partial_y A_z - \partial_z A_y = 0$$

To solve these, we can set $A_z=0$ and assume A_x and A_y are independent of z. Finally, we can take A_y linear in x and/or A_x linear in y. For example, we can have:

$$\vec{A}=(0,\, Bx,\, 0)$$

$$\vec{A}=(-By,\, 0,\, 0)$$

$$\vec{A}=\left(-\tfrac{1}{2}By,\, \tfrac{1}{2}Bx,\, 0\right) \quad (*)$$

Now, using any of these, we have:

$$H=\frac{p^2}{2m}-\frac{q}{m}\vec{p}\cdot A(x)+\frac{q^2}{2m}\vec{A}^2(x,t)$$

For example, using the last expression $(*)$, we get:

$$H=\frac{p^2}{2m}+\frac{q^2B^2}{8m}(x^2+y^2)+\frac{qB}{2m}(xp_y-yp_x)$$

This is a harmonic oscillator with a perturbation proportional to L_z!

(2)

We will use the first order time-dependent perturbation theory formula:

$$P_{|0\rangle \to |n\rangle} = \frac{1}{\hbar^2} \left| \int_0^t dt_1 \, e^{i\omega_{n0} t_1} H'_{n0}(t_1) \right|^2$$

Here $\omega_{n0} = \dfrac{E_n - E_0}{\hbar} = \dfrac{\hbar\omega(n+\frac{1}{2}) - \frac{1}{2}\hbar\omega}{\hbar} = \omega \cdot n$

$$H'_{n0}(t) = \langle n | \alpha x^2 \sin(ft) | 0 \rangle$$

$$= \alpha \sin(ft) \langle n | x^2 | 0 \rangle$$

$$= \alpha \sin(tf) \langle n | (a + a^\dagger)^2 | 0 \rangle \cdot \frac{\hbar}{m\omega}$$

We see that this vanishes for $n=1$, so there are no transitions to the $n=1$ state. For $n=2$, we have:

$$H'_{20}(t) = \alpha \sin(tf) \langle 2 | (a+a^\dagger)(a+a^\dagger) | 0 \rangle \frac{\hbar}{m\omega}$$

$$= \frac{\alpha \sqrt{2} \hbar}{m\omega} \sin(tf)$$

Then $P_{0\to 2} = \dfrac{2\alpha^2}{m^2\omega^2} \left| \int_0^t dt_1 \, e^{i\omega \cdot 2 \cdot t_1} \sin(t_1 f) \right|^2$

For $t = \frac{\pi}{f}$: $P_{0\to 2} = \dfrac{2\alpha^2}{m^2\omega^2} \left| \int_0^{\pi/f} dt_1 \, e^{i\omega \cdot 2 \cdot t_1} \sin(t_1 f) \right|^2$

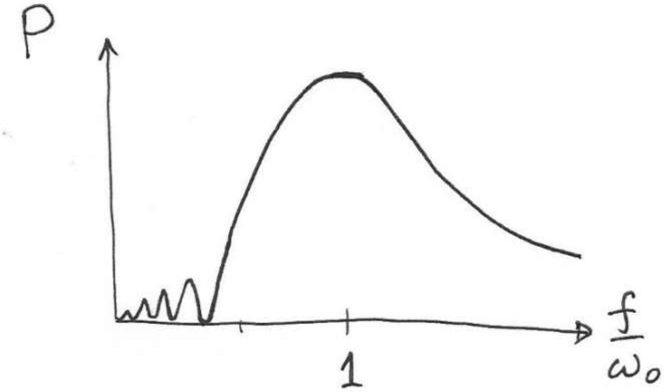
$$= \frac{2\alpha^2}{m^2} \cdot \frac{1}{\omega^2 f^2} \left| \int_0^{\pi} ds \, e^{2i\frac{\omega}{f}s} \sin(s) \right|^2$$

$$= \frac{8\alpha^2}{m^2} \frac{1}{\omega^2 f^2} \frac{\cos^2(\frac{\omega}{f}\pi)}{(4(\omega/f)^2 - 1)^2}$$

Defining $\omega_0 = 2\omega$, this is

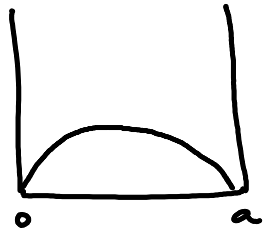
$$P_{0\to 2} = \frac{32\alpha^2}{m^2} \cdot \frac{1}{\omega_0^2 f^2} \frac{\cos^2\left(\frac{\omega_0}{f} \cdot \frac{\pi}{2}\right)}{\left(\left(\frac{\omega_0}{f}\right)^2 - 1\right)^2}$$

This looks like:



We see that this is peaked at $f = \omega_0$, but not very sharply; the peak would be much sharper if we left the perturbation on for a longer time.

(3) We have:



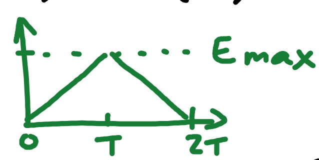
$$\psi_1(t=0) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)$$

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \qquad E_n = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$$

The probability that the particle will be found in the first excited state at time $t=2T$ is:

$$P(2T) = \frac{1}{\hbar^2}\left|\int_0^{2T} dt\, e^{i\omega_{21}t} H'_{21}(t)\right|^2$$

Here
$$H'_{21}(t) = qE(t)\,\langle\psi_2|x|\psi_1\rangle$$
$$= qE(t)\cdot\int_0^a dx\,\frac{2}{a}\cdot\sin\left(\frac{2\pi x}{a}\right)\cdot x\cdot\sin\left(\frac{\pi x}{a}\right)$$
$$= 2q\cdot E(t)\cdot a\int_0^1 d\hat{x}\,\sin(2\pi\hat{x})\cdot\hat{x}\cdot\sin(\pi\hat{x})$$
$$= -\frac{16}{9\pi^2}\cdot q\cdot a\cdot E(t)$$



So: $$P(2T) = \frac{256}{81\pi^4}\cdot\frac{q^2a^2}{\hbar^2}\cdot\left|\int_0^{2T} dt\; e^{i(E_2-E_1)t/\hbar}\, E(t)\right|^2$$

$$= \frac{256}{81\pi^4}\frac{q^2a^2}{\hbar^2}\left|\int_0^T dt\; e^{i\frac{3\hbar\pi^2}{2ma^2}t}\,\frac{E_{max}\cdot t}{T} + \int_T^{2T} dt\; e^{i\frac{3\hbar\pi^2}{2ma^2}t}\,\frac{E_{max}}{T}(2T-t)\right|^2$$

$$= \left(\frac{256\,q a^5 m^2 E}{81\pi^6\hbar^3 T}\right)^2 \sin^4\left(\frac{3}{4}\frac{T\hbar\pi^2}{ma^2}\right)$$

b) As we take $T\to\infty$, this transition probability goes to zero, consistent with the idea that the state should remain in the ground state for a very slow perturbation.