Problem set 10

Question 1 (Webwork due Tuesday)

a) Suppose we have a quantum system with a three-dimensional Hilbert space described by a Hamiltonian whose matrix representation in some basis is

$$E_0 \begin{pmatrix} 3 & 4 & 5 \\ 4 & 7 & 1 \\ 5 & 1 & 6 \end{pmatrix} . \tag{1}$$

Using the variational method and taking trial states to be the three basis states in the basis we are using, what is the best upper bound that we obtain?

b) For a particle in a delta function potential $V = C\delta(x)$ with $C > 0$, if we use a variational wavefunction $\psi(x) = A/(x^2 + b^2)$ (as in the example from the video), what is the best upper bound that we can place on the ground state energy?

Question 2 (Hand in Thursday via Canvas) Consider the one-dimensional problem of a particle of mass m moving in a potential $V(x) = \alpha|x|^3$.

a) Use dimensional analysis to determine the dependence of the ground state energy on α, $\hbar$, and m.

Use variational methods to put an upper bound on the numerical coefficient. Try to get the best (lowest) result you can.

Hint: for this part, you can set $\hbar = \alpha = m = 1$ since you already know how these come into the final answer. Another tip: in computing $\langle\psi|p^2|\psi\rangle$, you can use that this is the inner product of $p|\psi\rangle$ with itself (so you only have to work with the first derivative of your trial wavefunction).

There will be a prize for the best bound (you must report your trial wavefunction as well as the result)

Question 3: (Hand in Tuesday via Canvas, participation credit) This question is a warm-up for our discussion of time-dependent perturbation theory next week. It is very similar to the content of chapter 9.1 of Griffiths (maybe 10.1 in the new edition?). You can have a look there if you get stuck. Consider a system with time-independent Hamiltonian H_0, initially in some state $|\Psi(0)\rangle$ at $t = 0$. At time $t = 0$, we add to the Hamiltonian a time-dependent perturbation $H'(t)$.

a) Write down the Schrödinger equations that governs the future evolution of the state.

b)Let $|\psi_n\rangle$ be the energy eigenstates of H_0. Then we can write the state at $t = 0$ as

$$|\Psi(0)\rangle = \sum_n c_n |\psi_n\rangle \tag{2}$$

Without the perturbation, the state will evolve as

$$|\Psi(t)\rangle = \sum_n c_n e^{-\frac{iE_n t}{\hbar}} |\psi_n\rangle \tag{3}$$

With the perturbation, we can still expand the state at any time in terms of the eigenstates of H_0, but the time-dependence of the coefficients will be more complicated. Let's write this as

$$|\Psi(t)\rangle = \sum_n c_n(t) e^{-\frac{iE_n t}{\hbar}} |\psi_n\rangle \tag{4}$$

where now $c_n(t)$ are some unknown functions of time that we wish to determine (instead of just constants). By plugging this expression into the Schrödinger equation from part a) and taking the component of this equation corresponding to the basis element $|\psi_m\rangle$, show that

$$\frac{dc_m}{dt} = -\frac{i}{\hbar} \sum_n e^{\frac{i}{\hbar}(E_m - E_n)t} H'_{mn}(t) c_n(t) \tag{5}$$

where

$$H'_{mn}(t) = \langle \psi_m | H'(t) | \psi_n \rangle \ . \tag{6}$$

c) We'll discuss the solution of an equation like this in class. We'll work perturbatively in H'. As a warm-up, consider the following differential equation:

$$\frac{d}{dt} c(t) = f(t) c(t) \ . \tag{7}$$

Write the exact solution of this that gives $c(t)$ in terms of $c(0)$. Now expand the solution to show the terms with 0 and 1 power of $f(t)$.

d) Read through chapter 9.1 if you haven't already.

EXTRA: Identical particles (you won't be responsible for this, and I believe it was covered in Physics 304, but I'd strongly recommend reading this as a review)
We have talked about how when describing multipart systems, a basis for the full Hilbert space can be written in terms of bases $\{|n\rangle\}$ and $\{|N\rangle\}$ for the individual subsystems via the tensor product construction, where we have one basis element $|n\rangle \otimes |N\rangle$ for each pair.

There is an important modification to this when our quantum system is a system with two (or more) identical particles e.g. two electrons in a Helium atom. In this case, simply taking the tensor product Hilbert space is not quite right, since swapping the two particles gives a state that is actually physically equivalent to the original state. So it would be overcounting to include both $|n_1\rangle \otimes |n_2\rangle$ and $|n_2\rangle \otimes |n_1\rangle$ as distinct basis vectors for our Hilbert space.

An important result of quantum field theory (which is the most complete way to describe systems of multiple particles) is that for two identical particles with integer spins (e.g. alpha particles), the appropriate basis for the Hilbert space is the set of symmetrized states

$$\frac{1}{\sqrt{2}}(|n_1\rangle \otimes |n_2\rangle + |n_2\rangle \otimes |n_1\rangle) \ .$$

These states map to themselves under the exchange of the two particles; particles whose states have this property are called BOSONS. For multiparticle states, the basis elements are combinations that are invariant under the exchange of any two of the particles.

For identical particles with half-integer spins $1/2, 3/2, \ldots$ the appropriate basis of the Hilbert space it the set of antisymmetrized states

$$\frac{1}{\sqrt{2}}(|n_1\rangle \otimes |n_2\rangle - |n_2\rangle \otimes |n_1\rangle) \ .$$

These states transform as $|\Psi\rangle \to -|\Psi\rangle$ under the exchange of the two particles; particles whose states have this property are called FERMIONS. In this case, there are no basis elements with $n_1 = n_2$ since the expression above vanishes here: this leads to the *Pauli exclusion principle*, that identical particles with half-integer spins cannot be in the same state. You can read more about this in Griffiths 5.1.1.

These constraints on the symmetry properties of states have dynamical consequences for the energies and degeneracies of quantum systems, as you will see in the following exercise:

Consider two non-interacting particles in a 1D harmonic oscillator potential. Determine the lowest three energy levels and the corresponding degeneracies if the particles are a) identical bosons b) identical fermions. Write the states explicitly, remembering that the states must be symmetric/antisymmetic under exchange of the particles for bosons and fermions respectively.